

Properties of determinants

Math 4A – Scharlemann

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To any **square matrix** A , you can assign a real number, called the **determinant of A** (denoted $\det A$) that has nice properties:

- Represents the area/volume change of $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$
- $\det AB = \det A \cdot \det B$
- $\det A = 0 \iff A$ is **not** invertible
- $\det I_n = 1$

Definition complicated:

Pick any row (say the 1st row) or any column. Then

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

Here

$$C_{ij} = (-1)^{(i+j)} \det A_{ij}.$$

What's A_{ij} ?

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix} \implies A_{ij} = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}$$

The factor $(-1)^{(i+j)}$ can be pictured:

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The miracle: doesn't depend on row or column chosen.

Suppose A is an $n \times n$ matrix. Some properties:

- $\det A = \det A^T$. Easy example:

$$\det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \det \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

Idea: Can compute using cofactors on **rows** or **columns**.

- If a row (or column) of A is all zeros, then $\det A = 0$.

So,

$$\det \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 0 & 0 & 0 & 0 \\ 8 & 7 & 6 & 5 \end{bmatrix} = 0$$

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- We saw last time that for a square matrix A that is upper- or lower-**triangular**, $\det A$ is the product of its diagonal entries.

That is,

$$\det A = a_{11}a_{22} \cdots a_{nn}.$$

What is $\det \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 6 & 7 & 8 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$?

- A) $1 \cdot 2 \cdot 3 \cdot 4 = 4! = 24$
- B) $1 \cdot 0 \cdot 0 \cdot 0 = 0$
- C) $1 \cdot 6 \cdot 4 \cdot 5 = 120$
- D) What does triangular mean?

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- We learned last time that for A any square matrix, $\det A \neq 0 \iff A$ is invertible.

Is the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 6 & 7 & 8 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$ invertible?

- A) It is invertible since the determinant is $120 \neq 0$.
- B) It is **not** invertible since the determinant is $120 \neq 0$.
- C) It is invertible since the determinant is 0.
- D) It is **not** invertible since the determinant is 0.

Let A and B be two $n \times n$ matrices. Suppose B is obtained from A by multiplying a single row of A by r . Then $\det B = r \det A$.

Example:

$$\det \begin{bmatrix} 3 & 6 \\ 3 & 4 \end{bmatrix} = 3 \cdot \det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Reason: expand along the row that is multiplied by r , say the i^{th} row is multiplied by 5, so that $(B)_{ij} = b_{ij} = 5a_{ij}$, but **all cofactors are the same**:

$$\begin{aligned} \det B &= b_{i1}C_{i1} + b_{i2}C_{i2} + \cdots + b_{in}C_{in} = \\ &= 5a_{i1}C_{i1} + 5a_{i2}C_{i2} + \cdots + 5a_{in}C_{in} = \\ &= 5(a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}) = 5 \det A \end{aligned}$$

Theorem

Multiply row by scalar $\implies$ determinant multiplied by same scalar.

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Suppose A and B are $n \times n$ matrices and $B = 5A$. Then what is $\det B$?

- A) $5 \det A$
- B) $25 \det A$
- C) $5! \det A$
- D) $5^n \det A$
- E) $\frac{1}{5} \det A$

Discussion:

In general $\det rA = r^n \det A$, since there are n rows!

Example:

$$\begin{aligned}\det \left(3 \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \right) &= \det \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix} = 3 \cdot \det \begin{bmatrix} 1 & 2 \\ 3 & 12 \end{bmatrix} = 3 \cdot 3 \cdot \det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \\ &= 3^2(-2) = -18\end{aligned}$$

Theorem

Suppose that B is obtained from A by swapping two rows. Then $\det B = -\det A$.

Example:

$$\det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = -2 = -\det \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

To see this:

Switch **adjacent** rows and calculate determinant on one of them.

Each $C_{ij} = (-1)^{i+j} A_{ij}$ changes exactly by its sign $(-1)^{i+j}$.

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Exercise: **Any** switch comes from odd number of **adjacent** switches.

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What about the third type of elementary row operation: adding k times one row to another?

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad B = \begin{bmatrix} a & b \\ ka + c & kb + d \end{bmatrix}$$

How is $\det A$ related to $\det B$?

- A $\det A = \det B$.
- B $\det A = k(ad - bc) \det B$.
- C $\det A = k \det B$
- D I can't see any relation between them.

Discussion

In fact

$$\begin{aligned}\det B &= a(kb + d) - b(ka + c) \\ &= (akb + ad) - (bka + bc) = ad - bc = \det A\end{aligned}$$

So A is true.

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What happens if we do this to a 3 by 3 matrix?

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ ka_{11} + a_{21} & ka_{12} + a_{22} & ka_{13} + a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

How is $\det A$ related to $\det B$?

- A $\det A = \det B$.
- B $\det A = k \det B$
- C Enough already!

Discussion

Expand the two determinants via cofactors in any row that was **not** involved. (Row 3 in this example.) All the cofactors are 2 by 2 matrices, and the previous question applies.

Theorem

*Same is true for **any** size matrix. If you add a scalar times one row to another, the determinant is unchanged.*

This implies: If two rows of A are equal, then $\det A = 0$.

Two ways to see this:

- Subtract one row from the other & get a row of zeroes.
- Switch two rows: this changes the sign of the determinant, but not the matrix, hence not the determinant!

Can use these rules to calculate determinant of large matrices.

Example from **last time**:

$$A = \begin{bmatrix} 2 & 3 & -4 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{bmatrix} \implies \det A = -46$$

Row operation: subtract twice last row from first:

$$\begin{bmatrix} 0 & 5 & -14 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{bmatrix}$$

This type of row operation has **no effect** on det, so calculate:

$$\det A = \det \begin{bmatrix} 0 & 5 & -14 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{bmatrix} = 1 \cdot \det \begin{bmatrix} 5 & -14 \\ -4 & 2 \end{bmatrix} = 10 - 56 = -46$$

In general, start with $n \times n$ matrix A

- Row reduce A to **echelon form** B , which is upper triangular.
- Since B is upper triangular, $\det B$ is the product of diagonal elements. (Aside: when there are free variables bottom diagonal elements will be 0. This means $\det A = 0$ and you're done. If there are no free variables proceed...)
- Count number of row exchanges used, say k .
- List numbers by which you multiply rows: $r_1, r_2, \dots, r_\ell$
(Actually, you never **need** to do this operation.)
- Then

$$\det A = (-1)^k \frac{\det B}{r_1 \cdot r_2 \cdot \dots \cdot r_\ell}$$

Example:

$$A = \begin{bmatrix} 2 & 4 & 5 & 1 \\ 1 & 3 & 2 & -2 \\ -3 & -7 & -5 & 9 \\ -1 & -2 & 1 & 4 \end{bmatrix} \Rightarrow \text{row exchange} \begin{bmatrix} 1 & 3 & 2 & -2 \\ 2 & 4 & 5 & 1 \\ -3 & -7 & -5 & 9 \\ -1 & -2 & 1 & 4 \end{bmatrix}$$

$$\Rightarrow \text{column clear} \begin{bmatrix} 1 & 3 & 2 & -2 \\ 0 & -2 & 1 & 5 \\ 0 & 2 & 1 & 3 \\ 0 & 1 & 3 & 2 \end{bmatrix} \Rightarrow \text{row exchange} \begin{bmatrix} 1 & 3 & 2 & -2 \\ 0 & 1 & 3 & 2 \\ 0 & 2 & 1 & 3 \\ 0 & -2 & 1 & 5 \end{bmatrix}$$

$$\Rightarrow \text{column clear} \begin{bmatrix} 1 & 3 & 2 & -2 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & -5 & -1 \\ 0 & 0 & 7 & 9 \end{bmatrix} \Rightarrow \text{column clear} \begin{bmatrix} 1 & 3 & 2 & -2 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & -5 & -1 \\ 0 & 0 & 0 & \frac{38}{5} \end{bmatrix}$$

Hence:

$$\begin{aligned}\det A &= \det \begin{bmatrix} 2 & 4 & 5 & 1 \\ 1 & 3 & 2 & -2 \\ -3 & -7 & -5 & 9 \\ -1 & -2 & 1 & 4 \end{bmatrix} = (-1)^2 \det \begin{bmatrix} 1 & 3 & 2 & -2 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & -5 & -1 \\ 0 & 0 & 0 & \frac{38}{5} \end{bmatrix} \\ &= (-1)^2 \cdot 1 \cdot 1 \cdot (-5) \cdot \frac{38}{5} = -38\end{aligned}$$

Note: Could have stopped earlier

$$\det A = (-1)^2 \det \begin{bmatrix} 1 & 3 & 2 & -2 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & -5 & -1 \\ 0 & 0 & 7 & 9 \end{bmatrix} = \det \begin{bmatrix} -5 & -1 \\ 7 & 9 \end{bmatrix} = -45 + 7 = -38$$