

Vector spaces

Math 4A – Scharlemann

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CELL PHONES OFF

Much that we've learned doesn't require "vectors" to be in $\mathbb{R}^n$.
Can use our results elsewhere, if we hit the **abstract** button.

Example: Functions $f : \mathbb{R} \rightarrow \mathbb{R}$ behave a lot like vectors in $\mathbb{R}^n$:

- Can **add** two functions f and g : $f + g$ is the function

$$(f + g)(t) = f(t) + g(t) \quad \text{for all } t \in \mathbb{R}$$

Example: Let $f(t) = t^2$, $g(t) = e^t$. Then

$$(f + g)(t) = t^2 + e^t.$$

- Can **multiply** a function f by a scalar c : cf is the function

$$(cf)(t) = c(f(t)) \quad \text{for all } t \in \mathbb{R}$$

Example: Let $f(t) = \sin(t)$. Then $(5f)(t) = 5\sin(t)$.

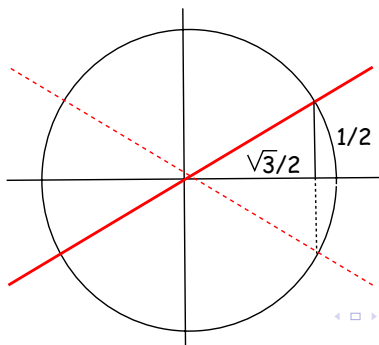
Then a **linear combination** of the functions $\{\sin t, \cos t\}$ is any

$$a \sin t + b \cos t \quad \text{for specific } a, b \in \mathbb{R} \text{ and all } t \in \mathbb{R}$$

iClicker question

Suppose $f(t) = 4 \cos t - \pi \sin t$. What is $f(\pi/6)$?

- A) $\sqrt{3} - \pi$
- B) $2\sqrt{3} - \pi$
- C) $\sqrt{3} - \frac{\pi}{2}$
- D) $2\sqrt{3} - \frac{\pi}{2}$
- E) We should know this?



Special case: Let P_n be all polynomials of degree at most n .
(These can be viewed as functions $\mathbb{R} \rightarrow \mathbb{R}$.)

Example: P_5 is every function we can write as

$$f(t) = a_5 t^5 + a_4 t^4 + a_3 t^3 + a_2 t^2 + a_1 t^1 + a_0$$

Can **add** any two of these or **multiply** any by a scalar.

Example: if

$$f(t) = 3t^5 - 2t^4 + t^3 + 4t^2 + t^1 - 3$$

then

$$5f(t) = 15t^5 - 10t^4 + 5t^3 + 20t^2 + 5t^1 - 15$$

Random thought: doesn't P_5 look a lot like R^6 ...?

iClicker question

Suppose $f(t) = t^5 + t^3 + 1$. $g(t) = -t^5 + t^3 - 1$.

What is $(f + g)(2)$?

- A) 4
- B) 6
- C) 8
- D) 16
- E) The function $f + g$ is not in P_5 .

Example from Math 4B:

Let V be all differentiable functions $f : \mathbb{R} \rightarrow \mathbb{R}$ so that

$$f'(t) = f(t) \quad \text{for all } t \in \mathbb{R}$$

Example: Let $f(t) = e^t$ Then [for all $t \in \mathbb{R}$]

$$f'(t) = e^t \implies f'(t) = f(t) \implies e^t \in V.$$

Suppose $f, g \in V$. Then [for all $t \in \mathbb{R}$]

$$(f+g)'(t) = f'(t) + g'(t) = f(t) + g(t) = (f+g)(t) \implies (f+g) \in V$$

Similarly, for any $c \in \mathbb{R}$, $cf \in V$.

Upshot: can add and scalar multiply in V just as we can in $\mathbb{R}^n$.

iClicker question

V is all differentiable functions $f : \mathbb{R} \rightarrow \mathbb{R}$ so that

$$f'(t) = f(t) \quad \text{for all } t \in \mathbb{R}.$$

We saw that $f(t) = e^t$ is an example of a function in V .
What's another example?

- A) $g(t) = e^t + 5$
- B) $g(t) = 5e^t$
- C) $g(t) = e^{5t}$
- D) $g(t) = \sin(t)$
- E) There are no other examples.

To capture all these situations define a **vector space** to be a set $\mathbb{V}$ so that for any $\vec{x}, \vec{y}, \vec{z} \in \mathbb{V}$ and $c, t \in \mathbb{R}$

- **addition**: $\vec{x} + \vec{y}$ is defined, and
- **scalar multiplication**: $c\vec{x}$ is defined.

Moreover:

V1 $\vec{x} + \vec{y} \in \mathbb{V}$

V2 $\vec{x} + \vec{y} = \vec{y} + \vec{x}$

V3 $(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$

V4 There is an element $\vec{0} \in \mathbb{V}$ so that $\vec{x} + \vec{0} = \vec{x} = \vec{0} + \vec{x}$

V5 For each $\vec{x} \in \mathbb{V}$ there is a $-\vec{x} \in \mathbb{V}$ so that $\vec{x} + (-\vec{x}) = \vec{0}$

V6 $c\vec{x} \in \mathbb{V}$

V7 $c(\vec{x} + \vec{y}) = c\vec{x} + c\vec{y}$

V8 $(c + t)\vec{x} = c\vec{x} + t\vec{x}$

V9 $c(t\vec{x}) = (ct)\vec{x}$

V10 $1\vec{x} = \vec{x}$

iClicker question

Consider all polynomials of the form

$$p(t) = t^2 + a$$

- A) This is a vector space because it is the same as P_2
- B) It is not a vector space because scalar multiplication is not defined
- C) It is not a vector space because even if $p(t)$ is such a polynomial, $cp(t)$ may not be.
- D) It is not a vector space because addition is not defined
- E) It is not a vector space because even if p and q are such polynomials, $p + q$ may not be.

Some easy consequences of being a vector space:

Theorem

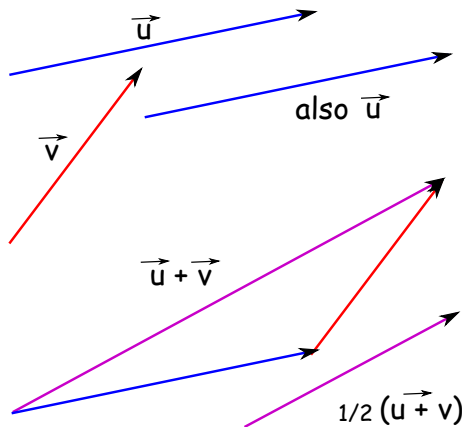
Let $\mathbb{V}$ be any vector space. Then

- $0\vec{x} = \vec{0}$ for all $\vec{x} \in \mathbb{V}$,
- $(-1)\vec{x} = -\vec{x}$ for all $\vec{x} \in \mathbb{V}$, and
- $t\vec{0} = \vec{0}$ for all $t \in \mathbb{R}$.

Sample proofs: Take any $\vec{x} \in \mathbb{V}$. Then:

- $\vec{0} = 0\vec{x} + (-\vec{0}\vec{x}) = (0 + 0)\vec{x} + (-\vec{0}\vec{x}) = 0\vec{x} + 0\vec{x} + (-\vec{0}\vec{x}) = 0\vec{x} + \vec{0} = 0\vec{x}$
- $-\vec{x} = -\vec{x} + \vec{0} = -\vec{x} + 0\vec{x} = -\vec{x} + (1 + (-1))\vec{x} = -\vec{x} + 1\vec{x} + (-1)\vec{x} = -\vec{x} + \vec{x} + (-1)\vec{x} = \vec{0} + (-1)\vec{x} = (-1)\vec{x}$

Another vector space: Let $\mathbb{V}$ be arrows in $\mathbb{R}^3$, regarding two as “the same” if they have the same direction and the same length.



Addition: put end-to-end (parallelogram rule)

Multiplication: keep direction the same, multiply length.

An important source of examples:

Definition

Suppose that $\mathbb{V}$ is a vector space and $\mathbb{U} \subset \mathbb{V}$. That is, $\mathbb{U}$ is contained in $\mathbb{V}$. Suppose further

- $\vec{0} \in \mathbb{U}$,
- for all $\vec{x}, \vec{y} \in \mathbb{U}$, the sum $\vec{x} + \vec{y} \in \mathbb{U}$
- for all $c \in \mathbb{R}$ and $\vec{x} \in \mathbb{U}$, the scalar product $c\vec{x} \in \mathbb{U}$

Then $\mathbb{U}$ is a **subspace** of $\mathbb{V}$

Theorem

Any subspace of a vector space is itself a vector space.

Example: Take $\mathbb{V}$ to be the vector space of all functions $f : \mathbb{R} \rightarrow \mathbb{R}$. Let $\mathbb{U} \subset \mathbb{V}$ be the set of all differentiable functions $f : \mathbb{R} \rightarrow \mathbb{R}$ so that $f' = f$. Clearly the function $f(t) = 0$ is in $\mathbb{U}$; the other two properties have already been shown.

iClicker question

Recall that P_5 is the set of all polynomials of degree at most 5, $\mathbb{V}$ is the vector space of all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ and $\mathbb{U}$ is the set of all differentiable functions $f : \mathbb{R} \rightarrow \mathbb{R}$ so that $f' = f$.

- A) P_5 is a subspace of $\mathbb{V}$.
- B) P_5 is not a subspace of $\mathbb{U}$ because a polynomial is not differentiable.
- C) P_5 is not a subspace of $\mathbb{V}$ because there are p, q in P_5 so that $p + q$ is not in P_5 .
- D) P_5 is not a subspace of $\mathbb{V}$ because there is a polynomial p in P_5 and some scalar c so that cp is not in P_5 .
- E) My brain is full.

Example: Let A be an $m \times n$ matrix.
Let $\mathbb{U}$ be all $\vec{x} \in \mathbb{R}^n$ so that $A\vec{x} = \vec{0}$.

Claim: $\mathbb{U}$ is a **subspace** of $\mathbb{R}^n$. (This is called the **nullspace** of A .)

Proof: Need to check three things:

- $\vec{0} \in \mathbb{U}$ (since $A\vec{0} = \vec{0}$).
- For any $\vec{x}$ & $\vec{y} \in \mathbb{U}$,

$$A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = \vec{0} + \vec{0} = \vec{0}.$$

Hence $\vec{x} + \vec{y} \in \mathbb{U}$.

- Similarly, for any $c \in \mathbb{R}$ and $\vec{x} \in \mathbb{U}$,

$$A(c\vec{x}) = cA\vec{x} = c\vec{0} = \vec{0}.$$

Hence $c\vec{x} \in \mathbb{U}$.

non-Example: The set of $\vec{x} \in \mathbb{R}^n$ so that $A\vec{x} = \vec{e}_1$ doesn't satisfy **any** of these three criteria, so it is **not** a subspace.

non-Example Let $\mathbb{U} \subset \mathbb{R}^2$ be the set of all $\begin{bmatrix} x \\ y \end{bmatrix}$ so that $x \geq 0, y \geq 0$. Check if it's a subspace of $\mathbb{R}^2$:

- $\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in \mathbb{U}$ ✓

- Suppose $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{U}$ and $\begin{bmatrix} x' \\ y' \end{bmatrix} \in \mathbb{U}$. Then

$$x, x' \geq 0 \implies x + x' \geq 0 \text{ and } y, y' \geq 0 \implies y + y' \geq 0.$$

$$\text{Hence } \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x + x' \\ y + y' \end{bmatrix} \in \mathbb{U} \quad \checkmark$$

- But, sadly, although $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \in \mathbb{U}$, the scalar product

$$-1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \notin \mathbb{U}. \quad \text{☹}$$

Since we have shown some example **does not** satisfy one of the criteria, $\mathbb{U}$ is **not** a subspace.

Another non-Example: Let $\mathbb{U} \subset \mathbb{R}^2$ be all $\begin{bmatrix} x \\ y \end{bmatrix}$ so that $x \cdot y \geq 0$.

Check if it's a subspace of $\mathbb{R}^2$:

- First criterion: $\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in \mathbb{U}$ since $0 \cdot 0 \geq 0$. ✓

- **Third** criterion: Suppose $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{U}$. Then $x \cdot y \geq 0$.

Take any $c \in \mathbb{R}$ and consider $c \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ cy \end{bmatrix}$

Calculate: $cx \cdot cy = c^2(x \cdot y) \geq 0$, so $c \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{U}$. ✓

- But (exercise) show that the **second criterion** is **not** satisfied, so $\mathbb{U}$ is **not** a subspace. (Hint: Consider $\begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \mathbb{U}$)

Example: Suppose $\mathbb{V}$ is any vector space and $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k \in \mathbb{V}$.

Definition

$\text{Span}\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k\}$ is the set of all elements of $\mathbb{V}$ that can be written as a **linear combination** of $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k\}$.

Example 1: In the vector space of all functions $f : \mathbb{R} \rightarrow \mathbb{R}$, $\text{Span}\{\sin(t), \cos(t)\}$ is the set of all functions of the form $a \sin(t) + b \cos(t)$.

Example 2: In the vector space of all functions $f : \mathbb{R} \rightarrow \mathbb{R}$, $\text{Span}\{1, t, t^2, t^3\}$ is the set of all functions of the form $a_3 t^3 + a_2 t^2 + a_1 t + a_0$.

In other words, $\text{Span}\{1, t, t^2, t^3\} = P_3!$

Theorem

$\text{Span}\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k\}$ is always a subspace of $\mathbb{V}$

Let's show this for just two vectors: let $\mathbb{U} = \text{Span}\{\vec{v}, \vec{w}\}$.

Check the three criteria:

- $\vec{0} = 0\vec{v} + 0\vec{w}$ so $\vec{0}$ is linear combination, so $\vec{0} \in \mathbb{U}$.
- Suppose $\vec{x}, \vec{y} \in \mathbb{U}$. This means there are $a, b, c, d \in \mathbb{R}$ so that

$$\vec{x} = a\vec{v} + b\vec{w} \quad \vec{y} = c\vec{v} + d\vec{w}$$

This means $\vec{x} + \vec{y} = (a + c)\vec{v} + (b + d)\vec{w}$.

Hence $\vec{x} + \vec{y}$ is a linear combination, so $\vec{x} + \vec{y} \in \mathbb{U}$

- Suppose $\vec{x} \in \mathbb{U}$, so there are $a, b \in \mathbb{R}$ with

$$\vec{x} = a\vec{v} + b\vec{w}.$$

Then for any $c \in \mathbb{R}$,

$$c\vec{x} = c(a\vec{v} + b\vec{w}) = ca\vec{v} + cb\vec{w}.$$

Hence $c\vec{x}$ is a linear combination, so $c\vec{x} \in \mathbb{U}$.