

# Special subspaces & linear transformations

Math 4A – Scharlemann

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CELL PHONES OFF

## Midterm Friday!

### Must bring

- TA.RD.IS code (on Gauchospace)
- Seat assignment (on Gauchospace - same as last time)
- Convincing photo id
- 8.5" by 11" bluebook for written work. Bring some to **sell!**

### May bring

- **One** 3" x 5" card
- Headlamp

### Don't bring

- Calculator
- Telephone or other electronics

Last time: a **vector space** is a set  $\mathbb{V}$  so that for any  $\vec{x}, \vec{y}, \vec{z} \in \mathbb{V}$  and  $c, t \in \mathbb{R}$

- **addition**:  $\vec{x} + \vec{y}$  is defined, and
- **scalar multiplication**:  $c\vec{x}$  is defined.

Moreover:

$$\text{V1 } \vec{x} + \vec{y} \in \mathbb{V}$$

$$\text{V2 } \vec{x} + \vec{y} = \vec{y} + \vec{x}$$

$$\text{V3 } (\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$$

$$\text{V4 } \text{There is an element } \vec{0} \in \mathbb{V} \text{ so that } \vec{x} + \vec{0} = \vec{x} = \vec{0} + \vec{x}$$

$$\text{V5 } \text{For each } \vec{x} \in \mathbb{V} \text{ there is a } -\vec{x} \in \mathbb{V} \text{ so that } \vec{x} + (-\vec{x}) = \vec{0}$$

$$\text{V6 } c\vec{x} \in \mathbb{V}$$

$$\text{V7 } c(\vec{x} + \vec{y}) = c\vec{x} + c\vec{y}$$

$$\text{V8 } (c + t)\vec{x} = c\vec{x} + t\vec{x}$$

$$\text{V9 } c(t\vec{x}) = (ct)\vec{x}$$

$$\text{V10 } 1\vec{x} = \vec{x}$$

Premier example (but just one example): **our friend**  $\mathbb{R}^n$ :  $n$ -tuples of real numbers, usually written as an  $n \times 1$  (vertical) matrix.

An important source of examples:

Suppose that  $\mathbb{V}$  is a vector space and  $\mathbb{U} \subset \mathbb{V}$ . That is,  $\mathbb{U}$  is a bunch of elements of  $\mathbb{V}$ . If

- $\vec{0} \in \mathbb{U}$ ,
- for all  $\vec{x}, \vec{y} \in \mathbb{U}$ , the sum  $\vec{x} + \vec{y} \in \mathbb{U}$
- for all scalars  $c$  and  $\vec{x} \in \mathbb{U}$ , the scalar product  $c\vec{x} \in \mathbb{U}$

Then  $\mathbb{U}$  is a **subspace** of  $\mathbb{V}$ .

**Important property:**  $\mathbb{U}$  is automatically a vector space.

## iClicker question

Consider the  $4 \times 4$  matrix

$$A = \begin{bmatrix} 2 & 4 & 5 & 1 \\ 1 & 3 & 2 & -2 \\ -3 & -7 & -5 & 9 \\ -1 & -2 & 1 & 4 \end{bmatrix}$$

Let  $\mathcal{U} \subset \mathbb{R}^4$  be the set of all  $\vec{x}$  so that  $A\vec{x} = \vec{0}$ .  
Is  $\mathcal{U}$  a subspace of  $\mathbb{R}^4$ ?

- A) Yes.
- B) No: there are two vectors which are in  $\mathcal{U}$  but their sum isn't.
- C) No: there is a vector in  $\mathcal{U}$ , but a scalar multiple is not in  $\mathcal{U}$ .
- D) This is a hard question.

Let  $A$  be an  $m \times n$  matrix.

Consider **all**  $\vec{u} \in \mathbb{R}^n$  so that  $A\vec{u} = \vec{0}$ . Call it **Nul  $A$** .

Last time: **Nul  $A$**  is a **subspace** of  $\mathbb{R}^n$ .

To show this, need to check three things:

- $\vec{0} \in \text{Nul } A$  (since  $A\vec{0} = \vec{0}$ ).
- For any  $\vec{x}$  &  $\vec{y} \in \text{Nul } A$ ,

$$A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = \vec{0} + \vec{0} = \vec{0}.$$

Hence  $\vec{x} + \vec{y} \in \text{Nul } A$ .

- Similarly, for any  $c \in \mathbb{R}$  and  $\vec{x} \in \text{Nul } A$ ,

$$A(c\vec{x}) = cA\vec{x} = c\vec{0} = \vec{0}.$$

Hence  $c\vec{x} \in \text{Nul } A$ .

**Easy** question: Given  $\vec{u} \in \mathbb{R}^n$ , is  $\vec{u} \in \text{Nul } A$ ?

**Harder** question: Describe all of  $\text{Nul } A$ .

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Example: Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 5 & -2 \\ 3 & 8 & 13 & -3 \end{bmatrix} \quad \vec{u} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

Is  $\vec{u}$  in  $\text{Nul } A$ ?

**Answer:** Just calculate  $A\vec{u}$ :  $A\vec{u} = \vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff \vec{u} \in \text{Nul } A$ .

Here

$$A\vec{u} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 5 & -2 \\ 3 & 8 & 13 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 - 2 \cdot 2 + 3 \cdot 1 + 1 \cdot 0 \\ 1 \cdot 1 - 2 \cdot 3 + 5 \cdot 1 - 2 \cdot 0 \\ 1 \cdot 3 - 2 \cdot 8 + 13 \cdot 1 - 3 \cdot 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence  $\vec{u} \in \text{Nul } A$ .

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**Harder** question: Describe **all** of  $\text{Nul } A$ . Interpretation: Find  $\vec{u}_1, \dots, \vec{u}_k \in \text{Nul } A$  so that  $\text{Span}\{\vec{u}_1, \dots, \vec{u}_k\} = \text{Nul } A$

Fancy description of an old problem we know how to do:  
Solve the homogeneous system of linear equations given by  $A\vec{x} = \vec{0}$ .

**Step 1:** Convert  $A$  to reduced echelon form:

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 5 & -2 \\ 3 & 8 & 13 & -3 \end{bmatrix} \xrightarrow{1^{st} \text{ col.}} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & -3 \\ 0 & 2 & 4 & -6 \end{bmatrix} \xrightarrow{2^{nd} \text{ col.}} \begin{bmatrix} 1 & 0 & -1 & 7 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

**Step 2:** Identify the free variables. Here they are  $x_3, x_4$ .

**Step 3:** Convert to vector equation:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 - 7x_4 \\ -2x_3 + 3x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -7 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{Then } \text{Nul } A = \text{Span} \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

This spanning set is **efficient**: no proper subset spans  $\text{Nul } A$ .

## Clicker question

Consider the  $4 \times 4$  matrix

$$A = \begin{bmatrix} 2 & 4 & 5 & 1 \\ 1 & 3 & 2 & -2 \\ -3 & -7 & -5 & 9 \\ -1 & -2 & 1 & 4 \end{bmatrix}$$

Let  $\mathcal{U} \subset \mathbb{R}^4$  be the span of all four column vectors in  $A$ .  
Is  $\mathcal{U}$  a subspace of  $\mathbb{R}^4$ ?

- A) Yes.
- B) No: there are two vectors which are in  $\mathcal{U}$  but their sum isn't.
- C) No: there is a vector in  $\mathcal{U}$ , but a scalar multiple is not in  $\mathcal{U}$ .
- D) This is a hard question.

As before, let  $A$  be any  $m \times n$  matrix:

Consider **all** linear combinations of the column vectors of  $A$ . It's a subset of  $\mathbb{R}^m$ , denoted **Col  $A$** .

In other words, for

$$A = [\vec{a}_1 \quad \vec{a}_2 \quad \dots \quad \vec{a}_n]$$

$\text{Col } A$  is  $\text{Span}\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\} \subset \mathbb{R}^m$ .

**Easier description:**  $\vec{y} \in \text{Col } A \iff$  equation  $\vec{y} = A\vec{x}$  has solution:

$$\vec{y} \in \text{Col } A \iff \vec{y} = x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_n\vec{a}_n \iff \vec{y} = A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

**Claim:**  $\text{Col } A$  is a **subspace** of  $\mathbb{R}^m$ .

To show this, need to check three things:

- $\vec{0} \in \text{Col } A$

Obvious:  $\vec{0} = A\vec{0}$ .

- For any  $\vec{y}_1, \vec{y}_2 \in \text{Col } A$ ,  $\vec{y}_1 + \vec{y}_2 \in \text{Col } A$ . Here's the argument:

Since  $\vec{y}_1, \vec{y}_2 \in \text{Col } A$  there are  $\vec{x}_1, \vec{x}_2$  so that  $\vec{y}_1 = A\vec{x}_1$  and  $\vec{y}_2 = A\vec{x}_2$ . Then

$$\vec{y}_1 + \vec{y}_2 = A\vec{x}_1 + A\vec{x}_2 = A(\vec{x}_1 + \vec{x}_2).$$

So  $\vec{x}_1 + \vec{x}_2$  is a solution to the equation  $\vec{y}_1 + \vec{y}_2 = A\vec{x}$ .

Hence  $\vec{y}_1 + \vec{y}_2 \in \text{Col } A$ .

- For any  $\vec{y} \in \text{Col } A$  and scalar  $c$ ,  $c\vec{y} \in \text{Col } A$ .

Here's the argument:

Since  $\vec{y} \in \text{Col } A$  there is an  $\vec{x}$  so that  $\vec{y} = A\vec{x}$ . Then

$$c\vec{y} = cA\vec{x} = A(c\vec{x}).$$

Hence  $c\vec{y} \in \text{Col } A$ .

**Easy** question: Describe Col  $A$ .

Answer: We've just done it – it's  $\text{Span}\{\vec{a}_1, \dots, \vec{a}_n\}$ , the span of the column vectors of  $A$ .

**Example:**

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 5 & -2 \\ 3 & 8 & 13 & -3 \end{bmatrix} \implies \text{Col } A = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 13 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix} \right\}$$

But this is not necessarily an **efficient** description: a subset may span Col  $A$ . In this case, the first two vectors suffice:

$$\text{Col } A = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 8 \end{bmatrix} \right\}$$

**Harder** question: Given  $\vec{u} \in \mathbb{R}^m$ , is  $\vec{u} \in \text{Col } A$ .

Translation: Are there  $\{x_1, \dots, x_n\}$  so that

$$\vec{u} = x_1 \vec{a}_1 + \dots + x_n \vec{a}_n?$$

Equivalently:

Is there a vector  $\vec{x} \in \mathbb{R}^n$  so that  $A\vec{x} = \vec{u}$ ?

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**Solution:** Reduce **augmented** matrix  $[\vec{a}_1 \ \vec{a}_2 \ \dots \ \vec{a}_n \ \vec{u}]$  to echelon form and see if the equations are consistent:

Example: Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 5 & -2 \\ 3 & 8 & 13 & -3 \end{bmatrix} \quad \vec{u} = \begin{bmatrix} 3 \\ 4 \\ 11 \end{bmatrix}$$

Is  $\vec{u} \in \text{Col } A$ ?

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 3 \\ 1 & 3 & 5 & -2 & 4 \\ 3 & 8 & 13 & -3 & 11 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 1 & 3 \\ 0 & 1 & 2 & -3 & 1 \\ 0 & 2 & 4 & -6 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 1 & 3 \\ 0 & 1 & 2 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the last column is not a pivot column, the associated equations are consistent. So indeed  $\vec{u} \in \text{Col } A$ .

Additional payoff: First two columns are the pivot columns  $\implies$  first two **of the original columns** span  $\text{Col } A$ .

Everything can be abstracted:

Let  $T : \mathbb{V} \rightarrow \mathbb{W}$  be a **linear transformation** from the vector space  $\mathbb{V}$  to the vector space  $\mathbb{W}$ .

### Definition

- The **kernel** of  $T$  is the set of all  $\vec{v} \in \mathbb{V}$  so that  $T(\vec{v}) = \vec{0}$ .
- The **image** (or **range**) of  $T$  is the set of all  $\vec{w} \in \mathbb{W}$  so that there is some  $\vec{v} \in \mathbb{V}$  with  $T(\vec{v}) = \vec{w}$ .

It's easy to see that

- the kernel of  $T$  is a **subspace** of  $\mathbb{V}$  (so it's a vector space)
- if  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  comes from matrix  $A$ , then kernel is **Nul  $A$**
- the image of  $T$  is a **subspace** of  $\mathbb{W}$  (so it's a vector space)
- if  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  comes from matrix  $A$ , then image is **Col  $A$**

**Sample proof:** Will show that the kernel of  $T$  (call it  $\text{Ker } T \subset \mathbb{V}$ ) is a subspace of  $\mathbb{V}$ :

Three criteria:

- $T(\vec{0}) = \vec{0} \implies \vec{0} \in \text{Ker } T. \quad \checkmark$
- Suppose  $\vec{v}_1, \vec{v}_2 \in \text{Ker } T$ . This means that  $T(\vec{v}_1) = \vec{0} = T(\vec{v}_2)$ . Since  $T$  is a linear transformation,

$$T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2) = \vec{0} + \vec{0} = \vec{0}$$

Hence  $\vec{v}_1 + \vec{v}_2 \in \text{Ker } T. \quad \checkmark$

- Suppose  $\vec{v} \in \text{Ker } T$ . This means that  $T(\vec{v}) = \vec{0}$ . Pick any scalar  $c \in \mathbb{R}$ . Since  $T$  is a linear transformation,

$$T(c\vec{v}) = cT(\vec{v}) = c\vec{0} = \vec{0}$$

Hence  $c\vec{v} \in \text{Ker } T. \quad \checkmark$

## iClicker question

As before, let  $\mathbb{V}$  be the set of all functions  $\mathbb{R} \rightarrow \mathbb{R}$  and  $\mathbb{U}$  be the subspace of **differentiable** functions  $\mathbb{R} \rightarrow \mathbb{R}$ . Define  $D : \mathbb{U} \rightarrow \mathbb{V}$  by the rule  $D(f) = f'$ .

Which sentence below best describes the situation?

- A) The rule  $D$  is not a linear transformation because it is not well-defined:  $D(f)$  is represented by lots of functions that just differ by a constant.
- B) The rule  $D$  is not a linear transformation because there are  $f, g \in \mathbb{V}$  so that  $D(f + g) \neq D(f) + D(g)$ .
- C)  $D$  is a linear transformation whose kernel is the function  $f(t) = 0$ .
- D)  $D$  is a linear transformation whose kernel consists of all constant functions.