

Basis & Bases

Math 4A – Scharlemann

18 February 2015



CELL PHONES OFF

Midterm Friday!

Must bring

- TA.RD.IS code (on Gauchospace)
- Seat assignment (on Gauchospace - same as last time)
- Convincing photo id
- 8.5" by 11" bluebook for **clear written work**. Bring some to **sell**!

May bring

- **One** 3" x 5" card
- Headlamp

Don't bring

- Calculator
- Telephone or other electronics

iClicker question

Let $\mathbb{V}$ be any vector space and suppose $\vec{a}_1, \dots, \vec{a}_k$ are elements (vectors) in $\mathbb{V}$.

The set $\{\vec{a}_1, \dots, \vec{a}_k\}$ is **linearly independent** if

A) $c_1 = c_2 = \dots = c_k = 0$ is a **solution** to the equation

$$c_1 \vec{a}_1 + c_2 \vec{a}_2 + \dots + c_k \vec{a}_k = \vec{0}$$

B) $c_1 = c_2 = \dots = c_k = 0$ is **the only** solution to the equation

$$c_1 \vec{a}_1 + c_2 \vec{a}_2 + \dots + c_k \vec{a}_k = \vec{0}$$

C) For any vector $\vec{v} \in \mathbb{V}$ there are values of $c_1, c_2, \dots, c_k$ so that

$$c_1 \vec{a}_1 + c_2 \vec{a}_2 + \dots + c_k \vec{a}_k = \vec{v}$$

D) For the matrix $A = [\vec{a}_1, \dots, \vec{a}_k]$ the only solution to $A\vec{x} = \vec{0}$ is $\vec{0}$.

Discussion

A) $c_1 = c_2 = \dots = c_k = 0$ is a **solution** to the equation

$$c_1 \vec{a}_1 + c_2 \vec{a}_2 + \dots + c_k \vec{a}_k = \vec{0}$$

False! The statement is true of **any** set of vectors - nothing special about $\{\vec{a}_1, \dots, \vec{a}_k\}$.

C) For any vector $\vec{v} \in \mathbb{V}$ there are values of $c_1, c_2, \dots, c_k$ so that

$$c_1 \vec{a}_1 + c_2 \vec{a}_2 + \dots + c_k \vec{a}_k = \vec{v}$$

False! This is the statement that $\mathbb{V} = \text{Span}\{\vec{a}_1, \dots, \vec{a}_k\}$. That's not the same as saying that $\{\vec{a}_1, \dots, \vec{a}_k\}$ are linearly independent.

D) For the matrix $A = [\vec{a}_1, \dots, \vec{a}_k]$ the only solution to $A\vec{x} = \vec{0}$ is $\vec{0}$.

False! Let's think about the matrix A . What is its size...?

A **only defined** if $\mathbb{V} = \mathbb{R}^m$ (some m); if not, $[\vec{a}_1, \dots, \vec{a}_k]$ **meaningless**.

Let $\mathbb{V}$ be any vector space and suppose $\vec{v}_1, \dots, \vec{v}_k$ are elements (vectors) in $\mathbb{V}$. The set $\{\vec{v}_1, \dots, \vec{v}_k\}$ is **linearly independent** if the **only** solution to the equation

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_k \vec{v}_k = \vec{0}$$

is the solution

$$c_1 = c_2 = \dots = c_k = 0.$$

(We said this before in the case $\mathbb{V} = \mathbb{R}^n$)

Put another way:

The set $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly **dependent** if there is a non-trivial solution to the equation

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_k \vec{v}_k = \vec{0}$$

Example: Suppose one of the vectors, say $\vec{v}_1$, is a linear combination of the others (i. e. $\vec{v}_1 \in \text{Span}\{\vec{v}_2, \vec{v}_3, \dots, \vec{v}_k\}$):

$$\vec{v}_1 = a_2 \vec{v}_2 + a_3 \vec{v}_3 + \dots + a_k \vec{v}_k$$

Then

$$(-1)\vec{v}_1 + a_2 \vec{v}_2 + a_3 \vec{v}_3 + \dots + a_k \vec{v}_k = \vec{0},$$

a non-trivial solution, so the set is linearly **dependent**.

Conclusion: If one element of a set of vectors ($\vec{v}_1$ in this example) is a linear combination of the others, then the set of vectors is linearly dependent.

Example: Suppose the set is linearly dependent, so there is a non-trivial solution

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_k \vec{v}_k = \vec{0}$$

Since the solution is non-trivial, one of the coefficients, say $c_1 \neq 0$. Subtract $c_1 \vec{v}_1$ from both sides and get:

$$c_2 \vec{v}_2 + \cdots + c_k \vec{v}_k = -c_1 \vec{v}_1$$

Divide (and you can!) by $-c_1 \neq 0$ and get

$$-\frac{c_2}{c_1} \vec{v}_2 - \cdots - \frac{c_k}{c_1} \vec{v}_k = \vec{v}_1$$

So $\vec{v}_1$ is a linear combination of the others in the set.

Conclusion: If a set of vectors is linearly dependent, then one element ($\vec{v}_1$ in this example) is a linear combination of the others.

Here are the two conclusions:

Conclusion 1: If one element of a set of vectors is a linear combination of the others, then the set is linearly dependent.

Conclusion 2: If a set of vectors is linearly dependent, then one element is a linear combination of the others.

Combine:

Theorem

A set of vectors is linearly dependent if and only if some element is a linear combination of the others (i. e. some element is in the span of the others).

*Put another way: a set of vectors is linearly **independent** if and only if **no** element is a linear combination of the others (i. e. **no** element is in the span of the others).*

iClicker question

Let $\mathbb{V}$ be any vector space and suppose $\{\vec{a}_1, \dots, \vec{a}_k\} \subset \mathbb{V}$.

The set $\{\vec{a}_1, \dots, \vec{a}_k\}$ **spans** $\mathbb{V}$ if

A) For any $\vec{v} \in \mathbb{V}$ there are values of $c_1, c_2, \dots, c_k$ so that

$$c_1 \vec{a}_1 + c_2 \vec{a}_2 + \dots + c_k \vec{a}_k = \vec{v}$$

B) For any $\vec{v} \in \mathbb{V}$ there are **non-zero** values of $c_1, c_2, \dots, c_k$ so that

$$c_1 \vec{a}_1 + c_2 \vec{a}_2 + \dots + c_k \vec{a}_k = \vec{v}$$

C) For any $\vec{v} \in \mathbb{V}$ there is **at most** one set of values for $c_1, c_2, \dots, c_k$ so that

$$c_1 \vec{a}_1 + c_2 \vec{a}_2 + \dots + c_k \vec{a}_k = \vec{v}$$

D) For matrix $A = [\vec{a}_1, \dots, \vec{a}_k]$ and any $\vec{b}$, $A\vec{x} = \vec{b}$ has solution.

Discussion

B) For any $\vec{v} \in \mathbb{V}$ there are **non-zero** values of $c_1, c_2, \dots, c_k$ so that

$$c_1 \vec{a}_1 + c_2 \vec{a}_2 + \dots + c_k \vec{a}_k = \vec{v}$$

False! That each $c_i \neq 0$ is not required. In fact, if $\{\vec{a}_1, \dots, \vec{a}_k\}$ are **linearly independent**, you need all $c_i = 0$ when $\vec{v} = \vec{0}$!

That is, $c_1 \vec{a}_1 + c_2 \vec{a}_2 + \dots + c_k \vec{a}_k = \vec{0}$ only if each $c_i = 0$.

D) For matrix $A = [\vec{a}_1, \dots, \vec{a}_k]$ and any $\vec{b}$, $A\vec{x} = \vec{b}$ has solution.

False! Again, $[\vec{a}_1, \dots, \vec{a}_k]$ is meaningless, unless $\mathbb{V} = \mathbb{R}^m$, some m .

Here's a nice consequence:

Theorem

*A subset $\{\vec{v}_1, \dots, \vec{v}_k\} \subset \mathbb{V}$ that spans V is linearly **dependent** if and only if a **proper** subset spans V . In other words, a subset is linearly **independent** if and only if **no** proper subset spans V .*

Easy in one direction: Suppose $\vec{v}_1$ is removed from the set and $\{\vec{v}_2, \vec{v}_3, \dots, \vec{v}_k\}$ still span $\mathbb{V}$. Since $\vec{v}_1 \in \mathbb{V}$, $\vec{v}_1 \in \text{Span}\{\vec{v}_2, \vec{v}_3, \dots, \vec{v}_k\}$. So $\vec{v}_1$ is a linear combination of the others which we've just seen means $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \dots, \vec{v}_k\}$ is linearly dependent.

Other direction: Suppose $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \dots, \vec{v}_k\}$ is linearly dependent. We just saw that then one vector, say $\vec{v}_1$, is a linear combination of the others $\{\vec{v}_2, \vec{v}_3, \dots, \vec{v}_k\}$:

$$\vec{v}_1 = a_2 \vec{v}_2 + a_3 \vec{v}_3 + \dots + a_k \vec{v}_k.$$

We now show that the proper subset $\{\vec{v}_2, \vec{v}_3, \dots, \vec{v}_k\}$ also spans $\mathbb{V}$:

Take any $\vec{u} \in \mathbb{V} = \text{Span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$; then

$$\vec{u} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 + \dots + c_k \vec{v}_k.$$

Substitute from above for $\vec{v}_1$:

$$\begin{aligned} \vec{u} &= c_1(a_2 \vec{v}_2 + a_3 \vec{v}_3 + \dots + a_k \vec{v}_k) + c_2 \vec{v}_2 + c_3 \vec{v}_3 + \dots + c_k \vec{v}_k = \\ &= (c_1 a_2 + c_2) \vec{v}_2 + (c_1 a_3 + c_3) \vec{v}_3 + \dots + (c_1 a_k + c_k) \vec{v}_k. \end{aligned}$$

This shows that $\vec{u}$ is in $\text{Span}\{\vec{v}_2, \vec{v}_3, \dots, \vec{v}_k\}$.

Which leads to this **important** definition:

Definition

A subset $\{\vec{v}_1, \dots, \vec{v}_k\} \subset \mathbb{V}$ is a **basis** for V if and only if it is linearly independent **and** it spans V .

Mother of all examples: The collection $\{\vec{e}_1, \dots, \vec{e}_n\} \subset \mathbb{R}^n$ is a basis for the vector space $\mathbb{R}^n$.

Another useful set of examples:

Suppose $\mathbb{V}$ is any vector space and you happen to know that a finite set of vectors $\{\vec{v}_1, \dots, \vec{v}_n\}$ spans $\mathbb{V}$. Contained in $\{\vec{v}_1, \dots, \vec{v}_n\}$ is a smallest subset that spans - it will be linearly independent, hence a basis! That's worth repeating...

Theorem

Any finite spanning set of vectors contains a basis.

Argument: Suppose $\{\vec{v}_1, \dots, \vec{v}_n\} \subset \mathbb{V}$ spans $\mathbb{V}$

- Step 1: if $\{\vec{v}_1, \dots, \vec{v}_n\}$ is linearly **in**dependent, then it's a basis.
- Step 2: if it's linearly **de**pendent, some proper subset spans.
- Repeat the argument for that proper subset.

Process eventually stops, since $\{\vec{v}_1, \dots, \vec{v}_n\}$ is finite

Two important properties of a basis:

- No proper subset of a basis is a basis (it will no longer span).
- Adding an additional vector to a basis will no longer constitute a basis (no longer linearly independent).

A basis efficiently captures most information about a vector space.

Just shown:

Any finite set $\{\vec{v}_1, \dots, \vec{v}_n\} \subset \mathbb{V}$ of vectors that spans $\mathbb{V}$ **contains** a basis.

Is any **linearly independent** subset $\{\vec{v}_1, \dots, \vec{v}_k\} \subset \mathbb{V}$ **contained in** a basis?

- Step 1: if the set of vectors spans $\mathbb{V}$, it's a basis
- Step 2: If it doesn't span $\mathbb{V}$, add vector $\vec{v}_{k+1}$ from $\mathbb{V}$, chosen so that it is not in span of $\{\vec{v}_1, \dots, \vec{v}_k\}$. Then consider $\{\vec{v}_1, \dots, \vec{v}_k, \vec{v}_{k+1}\}$ which we know to still be linearly independent.
- Repeat

Problem: Process may not stop! But if eventually the set does span all of $\mathbb{V}$, it is then a basis.

iClicker question

Consider the set of 2-vectors

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \end{bmatrix} \right\}$$

Which is the most informative true statement:

- A) The set S is a basis for $\mathbb{R}^2$.
- B) Some proper subset of S is a basis for $\mathbb{R}^2$
- C) Each pair of vectors in S is a basis for $\mathbb{R}^2$
- D) Every proper subset of S is a basis for $\mathbb{R}^2$

Example: Given a linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ with matrix A , find a **basis** for **Ker T** .

Answer: We've done this! $\text{Ker } T = \text{Nul } A$, can be found via reduced echelon form.

Example from last time: Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 5 & -2 \\ 3 & 8 & 13 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 & 7 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\text{Nul } A$ spanned by one vector for each free variable:

Here the free variables are x_3, x_4 . First equation gives:

$$x_1 - x_3 + 7x_4 = 0 \implies x_1 = x_3 - 7x_4$$

Second equation gives:

$$x_2 + 2x_3 - 3x_4 = 0 \implies x_2 = -2x_3 + 3x_4$$

Combining: $\vec{x} \in \text{Nul } A \iff$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 - 7x_4 \\ -2x_3 + 3x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -7 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{Hence } \text{Ker } T = \text{Nul } A = \text{Span} \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

The bottom rows show that these vectors are linearly independent, so they are a **basis** for $\text{Nul } A$.

Example: Given a linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ with matrix A , find a **basis** for **image of T** .

Answer: Image of $T = \text{Col } A$.

Step 1: Reduce to echelon form.

Step 2: Identify the **pivot columns**

Step 3: These columns of the **original matrix!** are basis for $\text{Col } A$.

Same example: Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 5 & -2 \\ 3 & 8 & 13 & -3 \end{bmatrix} \xrightarrow{\text{row reduce}} \begin{bmatrix} 1 & 0 & -1 & 7 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot columns are first and second columns

First and second columns of **original A** span $\text{Col } A$:

$\text{Col } A = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 8 \end{bmatrix} \right\}$ Linearly independent, hence a **basis**.