

Vector coordinates with respect to a basis

Math 4A – Scharlemann

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Let $\mathbb{V}$ be any vector space and suppose $\{\vec{v}_1, \dots, \vec{v}_k\}$ is a **basis** for $\mathbb{V}$. This means:

- $\{\vec{v}_1, \dots, \vec{v}_k\}$ **spans** $\mathbb{V}$ and
- $\{\vec{v}_1, \dots, \vec{v}_k\}$ is a **linearly independent** set.

We saw last time that there are two equivalent ways of saying **both** these conditions at once:

- $\{\vec{v}_1, \dots, \vec{v}_k\}$ is a basis if and only if it is a **minimal** spanning set for $\mathbb{V}$. That is, $\{\vec{v}_1, \dots, \vec{v}_k\}$ spans $\mathbb{V}$ but no proper subset spans $\mathbb{V}$.
- $\{\vec{v}_1, \dots, \vec{v}_k\}$ is a basis if and only if it is a **maximal** linearly independent set. That is, $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly independent but if we add any other vector to the set, the set becomes linearly dependent.

iClicker question

Chris tells Mary that he has found three vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ that are a basis for a vector space $\mathbb{V}$. Mary says she doubts it: she doesn't believe that the set spans. She picks a vector $\vec{w}$ and asks Chris: show me that it is in the span of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

Chris says, "That's easy - in fact there are two answers:
 $\vec{w} = \vec{v}_1 + 2\vec{v}_2 + 3\vec{v}_3$ and $\vec{w} = 3\vec{v}_1 + 2\vec{v}_2 + \vec{v}_3$ ".

Suppose Chris is right. We can conclude:

- A) Chris called Mary's bluff: it's probably a basis.
- B) It can't be a basis because there is a vector not in $\text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$
- C) It can't be a basis because $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent.
- D) The information is not enough to decide if it's a basis.

Discussion

Given the information that

$$\vec{w} = \vec{v}_1 + 2\vec{v}_2 + 3\vec{v}_3$$

and

$$\vec{w} = 3\vec{v}_1 + 2\vec{v}_2 + \vec{v}_3$$

we can conclude that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly **dependent**, and so it can't be a basis.

Subtract the two expressions from each other:

$$\begin{aligned}\vec{0} &= \vec{w} - \vec{w} = (\vec{v}_1 + 2\vec{v}_2 + 3\vec{v}_3) - (3\vec{v}_1 + 2\vec{v}_2 + \vec{v}_3) = \\ &= (1 - 3)\vec{v}_1 + (3 - 1)\vec{v}_3 = -2\vec{v}_1 + 2\vec{v}_3.\end{aligned}$$

Since $-2\vec{v}_1 + 2\vec{v}_3 = \vec{0}$, the vectors are linearly **dependent**.

Continue to assume $\{\vec{v}_1, \dots, \vec{v}_k\}$ is a basis for $\mathbb{V}$.

Since $\{\vec{v}_1, \dots, \vec{v}_k\}$ spans $\mathbb{V}$:

For each vector $\vec{v} \in \mathbb{V}$ there is **at least** one way of picking $c_1, \dots, c_k \in \mathbb{R}$ so that

$$\vec{v} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_k \vec{v}_k$$

Since the set $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly independent:

For each vector $\vec{v} \in \mathbb{V}$ there is **at most** one way of picking $c_1, \dots, c_k \in \mathbb{R}$ so that

$$\vec{v} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_k \vec{v}_k$$

Why?

We have

$$\vec{v} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_k \vec{v}_k$$

Suppose $d_1, \dots, d_k \in \mathbb{R}$ is a possibly different set of numbers with

$$\vec{v} = d_1 \vec{v}_1 + d_2 \vec{v}_2 + \dots + d_k \vec{v}_k.$$

Then

$$\begin{aligned} \vec{0} &= \vec{v} - \vec{v} = (c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_k \vec{v}_k) - (d_1 \vec{v}_1 + d_2 \vec{v}_2 + \dots + d_k \vec{v}_k) \\ &= (c_1 - d_1) \vec{v}_1 + (c_2 - d_2) \vec{v}_2 + \dots + (c_k - d_k) \vec{v}_k \end{aligned}$$

Since $\{\vec{v}_1, \dots, \vec{v}_k\}$ is **linearly independent** this can only happen if

$$(c_1 - d_1) = (c_2 - d_2) = \dots = (c_k - d_k) = 0.$$

In other words, $c_1 = d_1, c_2 = d_2, \dots, c_k = d_k$.

We have seen:

Theorem

Suppose $\vec{v}_1, \dots, \vec{v}_k$ is a **basis** for $\mathbb{V}$. For any $\vec{v} \in \mathbb{V}$ there is **exactly** one set $c_1, \dots, c_k \in \mathbb{R}$ so that

$$\vec{v} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_k \vec{v}_k$$

Summary: Since the set of vectors spans, there is **at least** one set $c_1, \dots, c_k \in \mathbb{R}$ so that $\vec{v} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_k \vec{v}_k$. If we try out another set $d_1, \dots, d_k \in \mathbb{R}$ then

$$(c_1 - d_1)\vec{v}_1 + (c_2 - d_2)\vec{v}_2 + \dots + (c_k - d_k)\vec{v}_k = \vec{0}$$

so **linear independence** of $\vec{v}_1, \dots, \vec{v}_k$ says that then

$$(c_1 - d_1) = 0, (c_2 - d_2) = 0, \dots, (c_k - d_k) = 0.$$

In other words: each $d_i = c_i$ and the d_i are **not** different.

The list $d_1, \dots, d_k$ is the same as $c_1, \dots, c_k$

Terminology: For the **indexed** basis $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_k\}$, the unique list of real numbers $c_1, \dots, c_k$ so that

$$\vec{v} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_k \vec{v}_k$$

are the **coordinates** of $\vec{v}$ with respect to the basis $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_k\}$.
The vector

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} \in \mathbb{R}^k$$

is the **coordinate vector** of $\vec{v}$ with respect to $\mathcal{B}$; denoted $[\vec{v}]_{\mathcal{B}}$.

iClicker question

Suppose $\mathcal{B} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\} \subset \mathbb{R}^3$. What is $\begin{bmatrix} \pi \\ e \\ \sqrt{2} \end{bmatrix}_{\mathcal{B}}$?

- A) $\mathcal{B}$ is not a basis, because it doesn't span $\mathbb{R}^3$.
- B) $\mathcal{B}$ is not a basis, because $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ are not linearly independent.
- C) $\begin{bmatrix} \pi \\ e \\ \sqrt{2} \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} \pi \\ e \\ \sqrt{2} \end{bmatrix}$
- D) The information is not enough to decide

Since $\begin{bmatrix} \pi \\ e \\ \sqrt{2} \end{bmatrix} = \pi \vec{e}_1 + e \vec{e}_2 + \sqrt{2} \vec{e}_3$, answer C) is correct.

Simplest Example:

Take the standard basis $\mathcal{B} = \{\vec{e}_1, \dots, \vec{e}_k\}$ of $\mathbb{R}^k$ where $\vec{e}_i =$

Any $\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_k \end{bmatrix} \in \mathbb{R}^k$ can be written $u_1 \vec{e}_1 + \dots + u_k \vec{e}_k$ so $[\vec{u}]_{\mathcal{B}} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_k \end{bmatrix}$

Moral : the coordinate vector of $\vec{u} \in \mathbb{R}^k$, with respect to the standard basis $\{\vec{e}_1, \dots, \vec{e}_k\}$ of $\mathbb{R}^k$, is just the vector itself.

iClicker question

Suppose $\mathcal{B} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \subset \mathbb{R}^3$. What is $\begin{bmatrix} \pi \\ e \\ \sqrt{2} \end{bmatrix}_{\mathcal{B}}$?

A) This is the same question as before.

B) $\begin{bmatrix} \pi \\ e \\ \sqrt{2} \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} \sqrt{2} \\ e \\ \pi \end{bmatrix}$

C) $\begin{bmatrix} \pi \\ e \\ \sqrt{2} \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} \pi \\ e \\ \sqrt{2} \end{bmatrix}$ since $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} = \{e_1, e_2, e_3\}$

D) The information is not enough to decide

Since the **indexed** set $\mathcal{B}$ appears in order $\{e_3, e_2, e_1\}$ and

$$\begin{bmatrix} \pi \\ e \\ \sqrt{2} \end{bmatrix} = \sqrt{2}\vec{e}_3 + e\vec{e}_2 + \pi\vec{e}_1, \text{ answer B) is correct.}$$

How to find the coordinates of $\vec{u}$ for a **different basis** of $\mathbb{R}^k$?

Suppose $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_k\}$ is a different basis of $\mathbb{R}^k$.

Want to find $c_1, \dots, c_k$ so that

$$\vec{u} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_k \vec{v}_k$$

In other words, find the vector $\vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} \in \mathbb{R}^k$ so that, for the $k \times k$ matrix $P = [\vec{v}_1 \quad \vec{v}_2 \quad \dots \quad \vec{v}_k]$, we have

$$\vec{u} = [\vec{v}_1 \quad \vec{v}_2 \quad \dots \quad \vec{v}_k] \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} = P \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$$

How to find the coordinates of $\vec{u}$ for a **different basis** of $\mathbb{R}^k$?

Need to solve the vector equation $\vec{u} = P\vec{c}$, where P is the $k \times k$ matrix whose column vectors are the basis:

$$P = [\vec{v}_1 \quad \vec{v}_2 \quad \dots \quad \vec{v}_k] .$$

Answer: Since the columns of P are linearly independent, P is invertible, & the solution to $\vec{u} = P\vec{c}$ is simply:

$$\vec{c} = P^{-1}\vec{u}$$

Example: Suppose you are given this basis of $\mathbb{R}^2$:

$$\mathcal{B} = \left\{ \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$$

Find the coordinates of $\vec{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ with respect to this basis.

We have seen this is equivalent to solving the equation

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Can solve this as standard system of linear equations **or** note that

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}^{-1} = \frac{1}{-2} \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{bmatrix}$$

So

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

We have just shown that coordinates of $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ with respect to basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$$

are $c_1 = -3, c_2 = 2$. That is $\begin{bmatrix} 3 \\ 2 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$.

You'd be **nuts** not to check it:

$$-3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Suppose $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_k\}$ is a different basis of $\mathbb{R}^k$. The matrix

$$P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_k \end{bmatrix}.$$

is called the **change of coordinates** matrix from the basis $\mathcal{B}$ to the standard basis.

To find standard coordinates $\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{bmatrix}$ corresponding to $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix}_{\mathcal{B}}$ just

take the product:

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{bmatrix} = P \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix}_{\mathcal{B}}$$

To go the reverse direction, multiply by P^{-1} :

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{bmatrix} = P \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix}_B \iff \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix}_B = P^{-1} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{bmatrix}$$

If Q is the change of coordinates matrix from **another** basis B' to the standard basis, then $Q^{-1}P$ changes coordinates from B to B' :

$$Q \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_k \end{bmatrix}_{B'} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{bmatrix} = P \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix}_B \implies \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_k \end{bmatrix}_{B'} = Q^{-1}P \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix}_B$$

Final point: If an arbitrary vector space $\mathbb{V}$ has a basis $\mathcal{B}$ of k vectors, then there is a natural invertible linear transformation

$$T : \mathbb{V} \rightarrow \mathbb{R}^k$$

given by $T(\vec{v}) = [\vec{v}]_{\mathcal{B}}$.

T assigns to a vector in $\mathbb{V}$ its coordinate vector with respect to $\mathcal{B}$.

A vector space with basis of k vectors looks just like (“is isomorphic to”) $\mathbb{R}^k$.

Example: P_5 is isomorphic to $\mathbb{R}^6$:

$$a_5 t^5 + a_4 t^4 + a_3 t^3 + a_2 t^2 + a_1 t^1 + a_0 \leftrightarrow \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_5 \end{bmatrix}$$