

Dimension and Rank

Math 4A – Scharlemann

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CELL PHONES OFF

Recall two important properties of a basis:

- No proper subset of a basis is a basis (it will no longer span).
- Adding an additional vector to a basis will no longer constitute a basis (no longer linearly independent).

Second property can be strengthened:

Proposition

Suppose V has **some** basis $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ with n vectors. Then **any** set $\{\vec{u}_1, \dots, \vec{u}_p\}$ with **more than n vectors** is linearly dependent.

Consider **special case** $V = \mathbb{R}^n$ (which has basis $\mathcal{B} = \{\vec{e}_1, \dots, \vec{e}_n\}$).
We've already secretly shown that, if $p > n$, any $\{\vec{u}_1, \dots, \vec{u}_p\} \subset \mathbb{R}^n$
are linearly dependent:

Consider the $n \times p$ matrix $A = [\vec{u}_1 \ \dots \ \vec{u}_p]$. Since $p > n$ there
are free variables $\implies$ there is non-zero solution to $A\vec{c} = \vec{0}$.
In other words:

$$A\vec{c} = [\vec{u}_1 \ \dots \ \vec{u}_p] \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix} = c_1\vec{u}_1 + c_2\vec{u}_2 + \dots + c_p\vec{u}_p = \vec{0}$$

This means set $\{\vec{u}_1, \dots, \vec{u}_p\}$ is linearly dependent.

General case: We are given vectors $\{\vec{u}_1, \dots, \vec{u}_p\}$ and the information that V has some basis $\mathcal{B}$ with $n < p$ vectors.

Let A be matrix with j -th column $\vec{a}_j$, the coordinate vector $[\vec{u}_j]_{\mathcal{B}}$ for $\vec{u}_j$ wrt the basis $\mathcal{B}$. Again it's $n \times p$ matrix.

Just as in special case, since A is $n \times p$ matrix, there is non-zero solution $\vec{c}$ to equation $A\vec{c} = \vec{0}$.

Then the coordinate vector of $c_1\vec{u}_1 + \dots + c_p\vec{u}_p$ is

$$[c_1\vec{u}_1 + \dots + c_p\vec{u}_p]_{\mathcal{B}} = c_1[\vec{u}_1]_{\mathcal{B}} + \dots + c_p[\vec{u}_p]_{\mathcal{B}} =$$

$$c_1\vec{a}_1 + \dots + c_p\vec{a}_p = A\vec{c} = \vec{0}.$$

But the only vector with coordinates $\vec{0}$ is $\vec{0}$ itself, so $c_1\vec{u}_1 + \dots + c_p\vec{u}_p = \vec{0} \implies$ the vectors are linearly dependent.

iClicker question

Here are 4 vectors in the vector space P_3 :

$$\begin{array}{rcl} & t^2 + & 2t - 1 \\ 2t^3 + 5t^2 - & & 7t + 3 \\ & 3t^2 + & 6t + 2 \\ -2t^3 - 5t^2 + & & 4t - 2 \end{array}$$

Are they linearly independent?

- A) These polynomials are clearly linearly independent
- B) These polynomials are clearly linearly dependent since $4 > 3$.
- C) I need more than a minute to figure this one out.

Discussion

The natural basis of P_n has $n + 1$ elements. For example, P_3 has this natural basis: $\mathcal{B} = \{t^3, t^2, t^1, t^0\} = \{t^3, t^2, t, 1\}$.

Since we are given 4 vectors, the previous discussion does **not** apply. We do **not** have more vectors than basis elements.

So B) is wrong: $4 > 3$ is not relevant. (If instead I had listed 5 cubic polynomials, they would be linearly **dependent** since $5 > 4$.)

The coordinate matrix of these vectors with respect to $\mathcal{B}$ is:

$$\begin{bmatrix} 0 & 2 & 0 & -2 \\ 1 & 5 & 3 & -5 \\ 2 & -7 & 6 & 4 \\ -1 & 3 & 2 & -2 \end{bmatrix}$$

The columns are linearly **independent**, but it's not obvious.

Corollary

Two bases of the same vector space have the same number of vectors.

Argument: Suppose $\mathcal{B}$ is a basis for $\mathbb{V}$ with n vectors and $\mathcal{B}'$ is a basis for $\mathbb{V}$ with p vectors. Since $\mathcal{B}$ is a basis with n vectors, first proposition says that any set of vectors with more than n vectors is linearly dependent. Hence $p \leq n$.

Reverse the roles of $\mathcal{B}$ and $\mathcal{B}'$ in this argument and get $n \leq p$.
Hence $p = n$. □

If there are n vectors in some (hence every) basis of a vector space $\mathbb{V}$, then n is called the **dimension** of $\mathbb{V}$, written $\dim(\mathbb{V})$.

Example: $\mathbb{R}^n$ has basis $\vec{e}_1, \dots, \vec{e}_n$ so $\mathbb{R}^n$ is **n-dimensional**. Duh!

If a vector space $\mathbb{V}$ is not spanned by any finite number of vectors, then $\mathbb{V}$ is **infinite dimensional**,

Example: the vector space $\mathcal{F}$ of all continuous functions $\mathbb{R} \rightarrow \mathbb{R}$ is infinite dimensional.

Proof(for the experts): Suppose $\mathcal{F}$ were n dimensional, for some n . We will show that this leads to a contradiction.

Check that the $n + 1$ functions $\{1, x, x^2, x^3, \dots, x^n \in \mathcal{F}\}$ are linearly independent. (This is a fun exercise, best done using derivatives!)

This contradicts the proposition above: these are $n + 1$ linearly independent vectors in an n -dimensional space.

Some sample applications: Suppose $\dim(\mathbb{V}) = n$

- Any **linearly independent set** of n vectors in $\mathbb{V}$ is a basis.
- Any **spanning set** of n vectors in $\mathbb{V}$ is a basis.

First claim: if the linearly independent set doesn't span, then can add a vector outside the span to get $n + 1$ linearly independent vectors. Contradicts proposition.

Second claim: if the spanning set is not linearly independent, can keep deleting vectors, without affecting span, until they are linearly independent. But this gives a basis with fewer than n vectors. Contradicts proposition.

Application to **subspaces**:

Corollary

Suppose $\mathbb{V}$ is a vector space with $\dim \mathbb{V} = n$. Suppose $\mathbb{U} \subset \mathbb{V}$ is a **subspace** of $\mathbb{V}$. Then

- $\dim \mathbb{U} \leq n$
- $\dim \mathbb{U} = n \iff \mathbb{U} = \mathbb{V}$

First claim: Start with a non-zero vector in $\mathbb{U}$ and keep putting more vectors in, keeping the set linearly independent, until you can put in no more. Know the process must stop, since you can't find more than n of them in V . When it stops, the resulting set of vectors must span $\mathbb{U}$; hence it's a basis.

Second claim: Suppose $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n\}$ is a basis for $\mathbb{U}$. If there were a vector in $\mathbb{V}$ outside $\text{Span}\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n\}$, putting it in the set would give $n + 1$ linearly independent vectors in n -dimensional space $\mathbb{V}$, a contradiction.

iClicker question

Find the dimensions of $\text{Nul } A$, $\text{Col } A$ for matrix:

$$A = \begin{bmatrix} 1 & 0 & 9 & 0 & -2 \\ 0 & 1 & 2 & 0 & 5 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- A) $\dim \text{Nul } A = 3, \dim \text{Col } A = 2$
- B) $\dim \text{Nul } A = 4, \dim \text{Col } A = 5$
- C) $\dim \text{Nul } A = \dim \text{Col } A = 2$
- D) $\dim \text{Nul } A = 2, \dim \text{Col } A = 3$
- E) I need more than a minute to figure this one out.

Discussion

A is already in reduced echelon form, and that's a good start:

Rule 1: $\dim \text{Nul}$ is exactly the number of **free variables**.

Reason: We have seen that **any** choice of free variables exactly determines what the pivot variables need to be. For example, the first line

$$[1 \ 0 \ 9 \ 0 \ -2] \implies x_1 + 9x_3 - 2x_5 = 0 \implies x_1 = -9x_3 + 2x_5$$

This process leads to the observation that

$$\text{Nul } A = \text{Span} \left\{ \begin{bmatrix} -9 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

The green rows show that the vectors are **linearly independent**.

Discussion

Rule 2: $\dim \text{Col}$ is exactly the number of **pivot columns**.

Reason: Each non-pivot column is linear combination of pivot columns to its left, since (in **reduced echelon form**) the pivot columns are $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_k$.

For example

$$A = \begin{bmatrix} 1 & 0 & 9 & 0 & -2 \\ 0 & 1 & 2 & 0 & 5 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} & & 9 & & -2 \\ \vec{e}_1 & \vec{e}_2 & 2 & \vec{e}_3 & 5 \\ & & 0 & & 1 \\ & & 0 & & 0 \end{bmatrix}$$

Thus pivot columns span all of $\text{Col } A$; since the pivot columns are $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_k$ the pivot columns are also linearly independent. Hence the pivot columns form a basis.

iClicker question

Find the dimensions of $\text{Nul } B$, $\text{Col } B$ for matrix:

$$B = \begin{bmatrix} 1 & -6 & 9 & 0 & -2 \\ 0 & 1 & 2 & -4 & 5 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- A) $\dim \text{Nul } B = 3, \dim \text{Col } B = 2$
- B) $\dim \text{Nul } B = 4, \dim \text{Col } B = 5$
- C) $\dim \text{Nul } B = \dim \text{Col } B = 2$
- D) $\dim \text{Nul } B = 2, \dim \text{Col } B = 3$
- E) I need more than a minute to figure this one out.

Discussion

This matrix is only in **echelon** form, but can see the number of pivot and non-pivot columns.

Row operations convert echelon form to **reduced echelon** form without changing the null space

Hence for **any** matrix B , $\dim \text{Nul } B$ is the number of free variables.

What about $\text{Col } B$? We have seen that **after row operations** to reduced echelon form, **all** columns are linear combinations of pivot columns, so pivot columns span the column space. But this fact won't be changed by row operations! Hence it is still true back in B that the pivot columns span $\text{Col } B$.

Summary: For **any** matrix B , the rules above hold:

$\dim \text{Nul } B$ is the number of non-pivot columns (i. e. number of free variables)

$\dim \text{Col } B$ is the number of pivot columns.

For $n \times p$ matrix, $\dim \text{Col } A + \dim \text{Nul } A = p$.

Warnings

- Need echelon form to tell which are pivot which not..
- Pivot columns in the **original matrix** B form basis for $\text{Col } B$.

So a basis for the column space of B above is:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \\ 5 \\ 0 \end{bmatrix} \right\}$$

Example: Consider the matrix A , which row reduces to the echelon form matrix B :

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 & 1 & 2 \\ 2 & 5 & 5 & 6 & 4 & 5 \\ 3 & 7 & 6 & 11 & 6 & 9 \\ 1 & 5 & 10 & 8 & 9 & 9 \\ 2 & 6 & 8 & 11 & 9 & 12 \end{bmatrix}; \quad B = \begin{bmatrix} 1 & 2 & 1 & 3 & 1 & 2 \\ 0 & 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Question: Are the column vectors of A linearly independent?

Answer: No, just looking at A : there are more vectors than dimension.

Question: What is $\dim \text{Col } A$?

Answer: Number of pivot columns = 3.

Question: What is $\dim \text{Nul } A$?

Answer: $6 - \dim \text{Col } A = 6 - 3 = 3$.

Question: Find a basis for $\text{Col } A$.

Answer: Pivot columns in B are first, second and fourth. Hence basis for $\text{Col } A$ are the first, second and fourth columns of A :

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 7 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 11 \\ 8 \\ 11 \end{bmatrix} \right\}$$

Remark 1: Finding basis for $\text{Nul } A$ is a bit more complicated, but we've done it for simpler examples.

Remark 2: $\dim \text{Col } A$ is called the **rank** of A ; it's the number of pivot columns. $\text{Rank } A$ is also the dimension of the **row** space of A .