

Eigenstuff

Math 4A – Scharlemann

27 February 2015



CELL PHONES OFF

A linear transformation T may be complicated, but look very simple on some vectors:

Definition

Let $T : \mathbb{V} \rightarrow \mathbb{V}$ be a linear transformation. Suppose for some $\lambda \in \mathbb{R}$ and $\vec{v} \neq 0 \in \mathbb{V}$,

$$T(\vec{v}) = \lambda \vec{v}$$

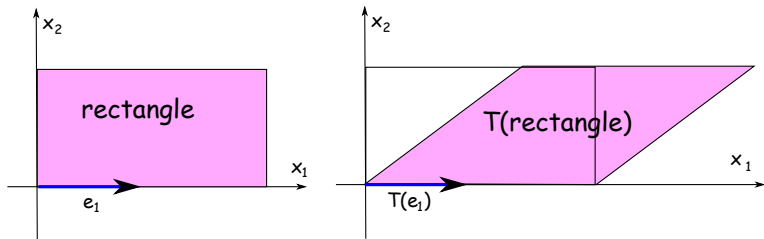
Then $\vec{v}$ is an **eigenvector** for T and λ is its **eigenvalue**.

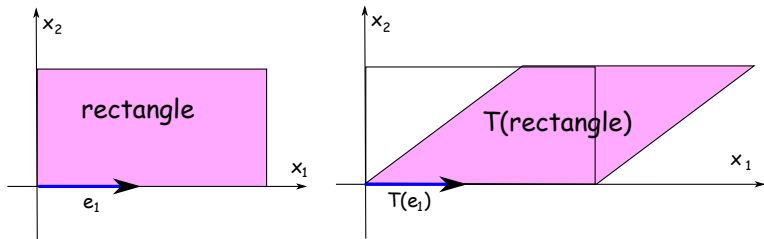
Thinking geometrically, $\vec{v}$ has the special property that its **direction** is the same, but its **size** is scaled by $\lambda \in \mathbb{R}$.

Addendum: When $\lambda < 0$ the direction of $\vec{v}$ is actually **reversed**.

$$T(\mu \vec{v}) = \mu T(\vec{v}) = \mu \lambda \vec{v} = \lambda(\mu \vec{v})$$

Example 1: Consider this shear $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$





Its matrix is $A = \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 1 \end{bmatrix}$.

Notice the horizontal vector $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ stays exactly the same.

Since

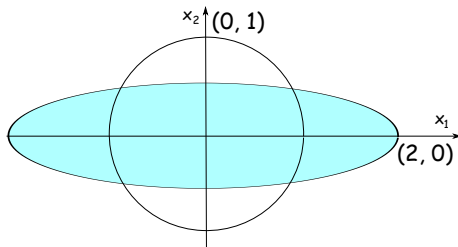
$$A\vec{e}_1 = \vec{e}_1 = 1 \cdot \vec{e}_1$$

the vector $\vec{e}_1$ is an **eigenvector** for T with **eigenvalue** 1.

Therefore all non-zero vectors along the x_1 -axis (i. e. all horizontal vectors) are eigenvectors with eigenvalue 1.

iClicker Question

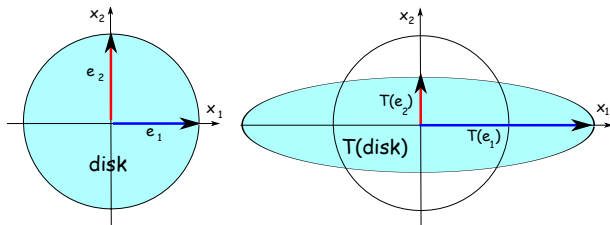
Recall the linear transformation that turns the unit circle into the ellipse, which we called a “scaling” linear transformation.



- A) There are no eigenvectors
- B) There are eigenvectors, and they all lie on a single line.
- C) There are eigenvectors, and they all lie on exactly two lines
- D) There are eigenvectors; they lie on three or more lines.
- E) We cannot tell from the given information.

Discussion

Example 2: Scaling: matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$.



Notice the horizontal $\vec{e}_1$ is doubled and the vertical $\vec{e}_2$ is half-sized.
Since

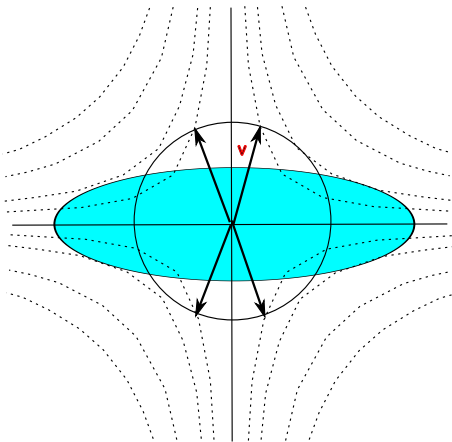
$$A\vec{e}_1 = 2 \cdot \vec{e}_1; \quad \text{and} \quad A\vec{e}_2 = \frac{1}{2} \cdot \vec{e}_2$$

$\vec{e}_1$ is eigenvector for T with eigenvalue 2;

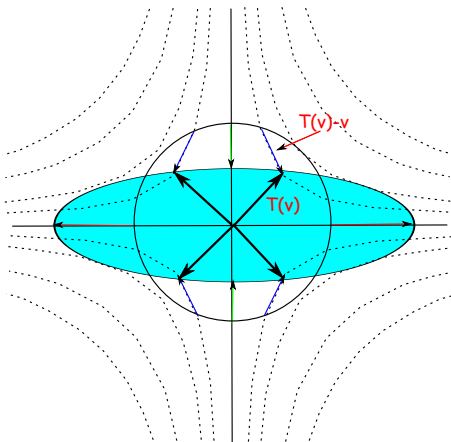
$\vec{e}_2$ is eigenvector for T with eigenvalue $\frac{1}{2}$.

$\vec{e}_1 + \vec{e}_2$ is not an eigenvector: $T(\vec{e}_1 + \vec{e}_2) = 2\vec{e}_1 + \frac{1}{2}\vec{e}_2$.

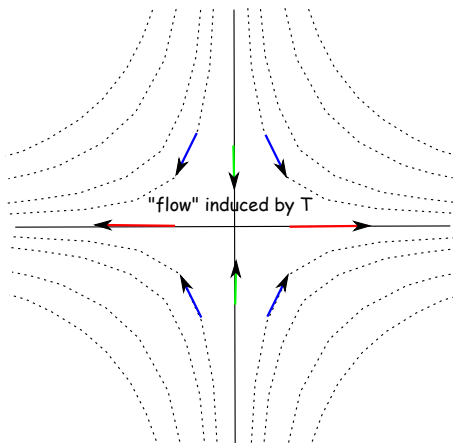
How non-eigenvectors change under scaling:



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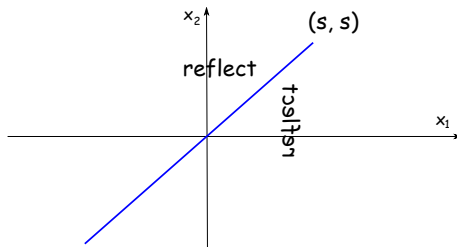


How non-eigenvectors change under scaling:



iClicker Question

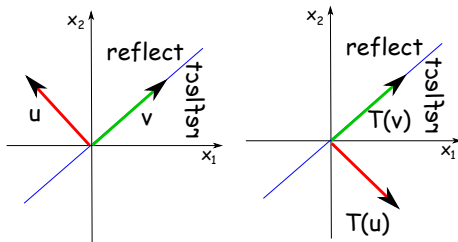
Recall the linear transformation that reflects across a line (say the line $x_1 = x_2$).



- A) There are no eigenvectors
- B) There are eigenvectors, and they all lie on a single line.
- C) There are eigenvectors, and they all lie on exactly two lines
- D) There are eigenvectors; they lie on three or more lines.
- E) We cannot tell from the given information.

Discussion

Example 3: Reflection: matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.



Notice $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ stays same and $\vec{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ points opposite.

Since

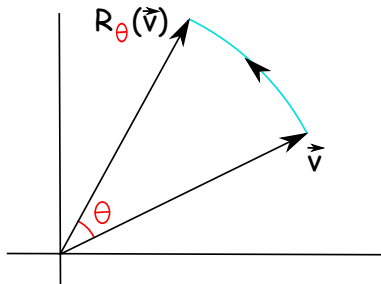
$$A\vec{v} = 1 \cdot \vec{v}; \quad \text{and} \quad A\vec{u} = -1 \cdot \vec{u}$$

$\vec{v}$ is eigenvector for T with eigenvalue $+1$;

$\vec{u}$ is eigenvector for T with eigenvalue -1 .

iClicker Question

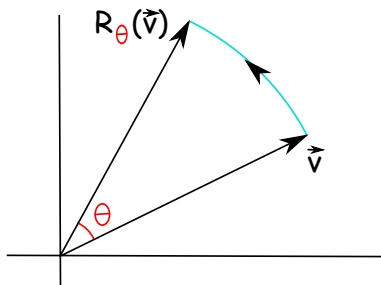
Recall the linear transformation that rotates around $\vec{0}$:



- A) There are no eigenvectors
- B) There are eigenvectors, and they all lie on a single line.
- C) There are eigenvectors, and they all lie on exactly two lines
- D) There are eigenvectors; they lie on three or more lines.
- E) We cannot tell from the given information.

Discussion

Example 4: Rotation: matrix $A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$.



Every vector is moved to a new direction, so **no eigenvectors!**
(Hence no eigenvalues either).

Suppose you know an **eigenvalue** λ of an $n \times n$ matrix A .

Theorem

The set E_λ of **eigenvectors** for T with eigenvalue λ (together with $\vec{0}$) is a **subspace** of $\mathbb{R}^n$. (Called the **eigenspace** for λ .)

Let's check: Suppose $\vec{v}, \vec{u} \in E_\lambda$ and $\mu \in \mathbb{R}$

- $\vec{0} \in E_\lambda$ ✓
- Since $T(\vec{v}) = \lambda\vec{v}$ and $T(\vec{u}) = \lambda\vec{u}$

$$T(\vec{v} + \vec{u}) = T(\vec{v}) + T(\vec{u}) = \lambda\vec{v} + \lambda\vec{u} = \lambda(\vec{v} + \vec{u})$$

Hence $(\vec{v} + \vec{u}) \in E_\lambda$. ✓

- For scalar μ have shown $T(\mu\vec{v}) = \lambda\mu\vec{v}$. So $\mu\vec{v} \in E_\lambda$. ✓

We secretly know how to find basis for an eigenspace of a linear transformation given by a matrix A , **once we know an eigenvalue**:

Suppose we are given a matrix A and an eigenvalue λ . Want to find all $\vec{v}$ so that $A\vec{v} = \lambda\vec{v}$:

$$A\vec{v} = \lambda\vec{v} \iff A\vec{v} = \lambda(I_n\vec{v}) = (\lambda I_n)\vec{v} \iff (A - \lambda I_n)\vec{v} = \vec{0}$$

Hence for matrix A , **eigenspace** E_λ with eigenvalue λ is the **nullspace** of $A - \lambda I_n$.

$$E_\lambda = \text{Nul}(A - \lambda I_n).$$

And we know how to find the basis of a nullspace.

Example: Your boss tells you that 4 is an eigenvalue for the matrix $A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$. She wants you to find all eigenvectors.

Answer:

$$A - 4I_n = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}$$

We must find $E_4 = \text{Nul} \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}$.

This matrix in echelon form is $\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$. Any solution to the equation

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$$

has the form $x_1 = x_2$, so the eigenvectors are all multiples of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Example: Your boss tells you that 3 is an eigenvalue for the matrix

$A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{bmatrix}$. She wants you to find all eigenvectors for this eigenvalue, in other words the eigenspace E_3 .

Answer:

$$A - 3I_n = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & -2 \\ 1 & 1 & -1 \end{bmatrix}$$

This matrix in echelon form is $B = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

So... find a basis for the nullspace (kernel) of B

Finding a basis for $\text{Ker } B$:

Any solution to the equation

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{0}$$

has the form

$$x_1 + x_2 - x_3 = 0 \implies x_1 = -x_2 + x_3,$$

so the eigenspace of A with eigenvalue 3 (which we have shown is the same thing as the kernel of B) is spanned by the vectors

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Green shows that vectors linearly independent, hence basis for E_3 .

Check: Are $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ eigenvectors of eigenvalue 3 for A ?

$$\begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

iClicker Question

Let

$$A = \begin{bmatrix} 4 & 1 & -1 & 7 \\ 0 & 5 & -2 & -4 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 13 \end{bmatrix}$$

How many different eigenvalues are there?

- A) There are no eigenvectors, so there are no eigenvalues.
- B) There is at least one eigenvalue.
- C) There are at least two eigenvalues.
- D) There are at least three eigenvalues.
- E) There are at least four eigenvalues.

Discussion

Easy calculation: Eigenvalues for a **triangular matrix** are the entries of the diagonal.

Reason: Suppose A is an upper triangular matrix and a_{kk} is one of the diagonal entries. Then $B = A - a_{kk}I_n$ is an upper triangular matrix with a 0 in the k^{th} diagonal spot.

So B is in echelon form with (at least) x_k as a free variable. Therefore it has non-trivial kernel.

But we have seen that the kernel of B is exactly the eigenspace for the eigenvalue a_{kk} .

Exercise: How do we know there aren't **other** eigenvalues?