

Echelon form

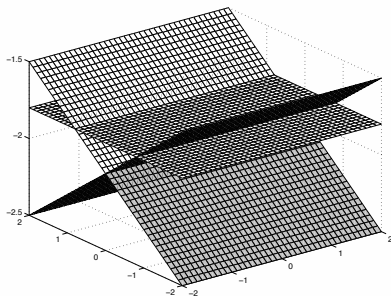
Math 4A – Scharlemann

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Mary plotted the solution set of each of 3 linear equations in 3 variables and got the 3 graphs shown. Most likely:

- A) The equations have a single solution.
- B) The equations are inconsistent.
- C) The equations are redundant.

A **system of linear equations** is a bunch of linear equations:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

Solution to system is list $s_1, \dots, s_n$ so that all m equations true.

Simplify the system until the set of solutions is obvious.

Solutions may not exist, may be unique, may be infinite.

First step: All the info is in the numbers a_{ij} , b .
So put them in a **matrix**:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

$m \times n$ **coefficient** matrix

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array} \right]$$

$m \times (n + 1)$ **augmented** matrix

These operations don't change the set of solutions for the system:

- Reorder the rows (interchange)
- Multiply a row by $c \neq 0$ (scale)
- Add a multiple of one row to another row (replacement)

Grand strategy:

Do this until the equations are easy to solve, for example:

Echelon form is pretty easy.

Reduced echelon form is easiest.

We will show how to get there and why this makes it easy.

Definition

A matrix is in **echelon form** if

- (1) all rows that are totally zero are at the bottom, and
- (2) the first non-zero entry in any row, called the **leading entry** is to the right of the leading entry of the previous row.

Example:

$$\begin{bmatrix} \$ & * & * & * & * & * \\ 0 & \$ & * & * & * & * \\ 0 & 0 & 0 & \$ & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

\$ = non-zero entry (leading entry or **pivot**)

* = any entry

Definition

A matrix is in **reduced** echelon form if

- (1) it is in echelon form,
- (2) all leading entries are 1, and
- (3) in a column with a leading 1, all other entries are zero.

Example:

$$\begin{bmatrix} 1 & 0 & * & 0 & * & * \\ 0 & 1 & * & 0 & * & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Theorem

Any matrix A reduces to a **unique** matrix in reduced echelon form.

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$$\begin{bmatrix} 0 & 1 & 2 & 4 & 0 & \pi & 0 \\ 0 & 0 & 3 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- A) Is in **echelon** form.
- B) Is not in echelon form since the left column is all 0.
- C) Is not in echelon form since the right column is all 0.
- D) Is not in echelon form since the 4 has only 0's below it.
- E) Is not in echelon form since it has a leading coefficient that is not 1

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$$\begin{bmatrix} 0 & 1 & 2 & 4 & 0 & \pi & 0 \\ 0 & 0 & 3 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- A) Is in **reduced** echelon form.
- B) Is not in reduced echelon form since the left column is all 0.
- C) Is not in reduced echelon form since the right column is all 0.
- D) Is not in reduced echelon form since 4 has only 0's below it.
- E) Is not in reduced echelon form since it has a leading coefficient that is not 1

Any matrix reduces to **echelon** form via these steps:

- (1) Ignore all totally zero columns on the left.
- (2) Interchange rows so that the top left entry in the first column is non-zero; call it the **pivot**.
- (3) Use replacement to zero out everything below pivot.
- (4) Move on to the submatrix south-east of the pivot .
- (5) Repeat the procedure for this submatrix.
- (6) Repeat until done (i. e. nothing but zeroes to the southeast)

Row operations take any **echelon** matrix to **reduced echelon** form:

(1) Scale each non-zero row by dividing by its leading entry

$$\begin{bmatrix} p_1 & * & * & * & * \\ 0 & p_2 & * & * & * \\ 0 & 0 & 0 & p_3 & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \frac{*}{p_1} & \frac{*}{p_1} & \frac{*}{p_1} & \frac{*}{p_1} \\ 0 & 1 & \frac{*}{p_2} & \frac{*}{p_2} & \frac{*}{p_2} \\ 0 & 0 & 0 & 1 & \frac{*}{p_3} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(2) Use replacement to zero out everything above pivots.

$$\begin{bmatrix} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{*s \text{ change}} \begin{bmatrix} 1 & 0 & * & 0 & * \\ 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example from last time:

$$\left[\begin{array}{ccc|c} 1 & -3 & -2 & 6 \\ 2 & -4 & -3 & 8 \\ -3 & 6 & 8 & -5 \end{array} \right] \xrightarrow{\text{replace}} \left[\begin{array}{ccc|c} 1 & -3 & -2 & 6 \\ 0 & 2 & 1 & -4 \\ 0 & -3 & 2 & 13 \end{array} \right] \xrightarrow{\text{replace}}$$

$$\left[\begin{array}{ccc|c} 1 & -3 & -2 & 6 \\ 0 & 2 & 1 & -4 \\ 0 & 0 & \frac{7}{2} & 7 \end{array} \right]$$

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We now have the matrix

$$\left[\begin{array}{ccc|c} 1 & -3 & -2 & 6 \\ 0 & 2 & 1 & -4 \\ 0 & 0 & \frac{7}{2} & 7 \end{array} \right]$$

This matrix

- A) Is in echelon form
- B) Is in reduced echelon form
- C) Both A & B
- D) Neither A nor B

Let's continue:

$$\left[\begin{array}{ccc|c} 1 & -3 & -2 & 6 \\ 0 & 2 & 1 & -4 \\ 0 & 0 & \frac{7}{2} & 7 \end{array} \right] \xrightarrow{\text{scale}} \left[\begin{array}{ccc|c} 1 & -3 & -2 & 6 \\ 0 & 1 & \frac{1}{2} & -2 \\ 0 & 0 & 1 & 2 \end{array} \right] \xrightarrow{\text{replace}}$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 0 & 10 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array} \right] \xrightarrow{\text{replace}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

This is now in **reduced** echelon form.

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We now have the matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

This means the solution is

- A) $(x_1, x_2, x_3) = (-1, 3, -2)$
- B) $(x_1, x_2, x_3) = (1, -3, 2)$
- C) $(x_1, x_2, x_3) = (-2, 3, -1)$
- D) $(x_1, x_2, x_3) = (2, -3, -1)$
- E) None of the above.

More sophisticated example

$$\left[\begin{array}{cccc|c} 0 & 0 & 6 & 0 & 12 \\ 2 & 2 & -1 & 6 & 4 \\ 4 & 4 & 1 & 10 & 13 \\ 8 & 8 & -1 & 26 & 23 \end{array} \right] \xrightarrow{\text{interchange}} \left[\begin{array}{cccc|c} 2 & 2 & -1 & 6 & 4 \\ 4 & 4 & 1 & 10 & 13 \\ 8 & 8 & -1 & 26 & 23 \\ 0 & 0 & 6 & 0 & 12 \end{array} \right] \xrightarrow{\text{replace}}$$

$$\left[\begin{array}{cccc|c} 2 & 2 & -1 & 6 & 4 \\ 0 & 0 & 3 & -2 & 5 \\ 0 & 0 & 3 & 2 & 7 \\ 0 & 0 & 6 & 0 & 12 \end{array} \right] \xrightarrow{\text{replace}} \left[\begin{array}{cccc|c} 2 & 2 & -1 & 6 & 4 \\ 0 & 0 & 3 & -2 & 5 \\ 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 4 & 2 \end{array} \right] \xrightarrow{\text{replace}}$$

$$\left[\begin{array}{cccc|c} 2 & 2 & -1 & 6 & 4 \\ 0 & 0 & 3 & -2 & 5 \\ 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

(Example continued...)

$$\left[\begin{array}{cccc|c} 2 & 2 & -1 & 6 & 4 \\ 0 & 0 & 3 & -2 & 5 \\ 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{\text{scale}} \left[\begin{array}{cccc|c} 1 & 1 & -\frac{1}{2} & 3 & 2 \\ 0 & 0 & 1 & -\frac{2}{3} & \frac{5}{3} \\ 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{\text{replace}}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{\text{replace}} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Variable(s) associated to non-pivot columns are free variables.

Why reduced echelon?

For matrix in this form, answers to our question are obvious:

- If **last** column of augmented matrix is pivot there is **no solution**:

$$\left[\begin{array}{cccc|c} 0 & 0 & \dots & 0 & 1 \end{array} \right] \Rightarrow 0x_1 + 0x_2 + \dots 0x_n = 1, \text{ absurd}$$

- If every column **but the last** is pivot there is a **unique solution**:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array} \right] \Rightarrow \begin{array}{l} x_1 = 1 \\ x_2 = -3 \\ x_3 = 2 \end{array}$$

- If there are **free variables** then infinitely many solutions.

How to find all solutions when there are free variables:

- Set free variables to anything
- Other variables are then completely determined.

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \Rightarrow \begin{array}{l} x_1 = \frac{3}{2} - x_2 \\ x_3 = 2 \\ x_4 = \frac{1}{2} \end{array}$$

Example with two free variables $\{x_2, x_4\}$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & \pi & 0 & \frac{3}{2} \\ 0 & 0 & 1 & e & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \Rightarrow$$

$$x_1 = \frac{3}{2} - x_2 - \pi x_4$$

$$x_3 = 2 - ex_4$$

$$x_5 = \frac{1}{2}$$