

Finding eigenvalues

Math 4A – Scharlemann

2 March 2015



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Last time: Find eigenvalues of triangular matrix

$$A = \begin{bmatrix} 4 & 1 & -1 & 7 \\ 0 & 5 & -2 & -4 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 13 \end{bmatrix}$$

Answer: λ is an eigenvalue $\iff A - \lambda I_4$ has non-trivial nullspace
 $\iff A - \lambda I_4$ has free variable $\iff \lambda$ is a diagonal entry.

Example: 2 is an eigenvalue. **Eigenspace** = **Null**($A - 2I_4$)

$$\begin{bmatrix} 2 & 1 & -1 & 7 \\ 0 & 3 & -2 & -4 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 11 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{6} & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ so } \text{Null}(A - 2I_4) = \text{Span} \left\{ \begin{bmatrix} \frac{1}{6} \\ \frac{2}{3} \\ 1 \\ 0 \end{bmatrix} \right\}$$

Check:

$$A \begin{bmatrix} 1 \\ 6 \\ 2 \\ 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 & 1 & -1 & 7 \\ 0 & 5 & -2 & -4 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 13 \end{bmatrix} \begin{bmatrix} 1 \\ 6 \\ 2 \\ 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{4}{6} + \frac{2}{3} - 1 \\ \frac{10}{3} - 2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{4}{3} \\ 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 6 \\ 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$$

So $\begin{bmatrix} 1 \\ 6 \\ 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$ is an eigenvector for A of eigenvalue 2.

iClicker Question

$$\begin{bmatrix} 4 & 1 & -1 & 7 \\ 0 & 0 & -2 & -4 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

How many different eigenvalues are there?

- A) There are no eigenvectors since the matrix is **not invertible**
- B) There is one eigenvalue.
- C) There are two eigenvalues.
- D) There are three eigenvalues.
- E) There are four eigenvalues because the matrix is triangular.

Discussion: The eigenvalues are 0, 2, 4.

Any non-zero vector in $\text{Nul } A$ is an eigenvector with eigenvalue 0:

Reason: if $\vec{v} \in \text{Nul } A$ then $A\vec{v} = \vec{0} = 0\vec{v}$.

Last time: Suppose A is $n \times n$ matrix and λ is an eigenvalue for A .
Can find corresponding **eigenspace**:

Eigenspace

The eigenspace of all vectors with eigenvalue λ is $\text{Nul}(A - \lambda I_n)$, a subspace of $\mathbb{R}^n$.

Example: Let

$$A = \begin{bmatrix} 4 & 1 & -1 & 7 \\ 0 & 5 & -2 & -4 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Eigenspace with eigenvalue 2 is the null space (a line) of

$$A - 2I_4 = \begin{bmatrix} 2 & 1 & -1 & 7 \\ 0 & 3 & -2 & -4 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Change to

$$B = \begin{bmatrix} 4 & 1 & -1 & 7 \\ 0 & 5 & -2 & -4 \\ 0 & 0 & 2 & \textcolor{red}{0} \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Eigenspace with eigenvalue 2 is now the null space of

$$B - 2I_4 = \begin{bmatrix} 2 & 1 & -1 & 7 \\ 0 & 3 & -2 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Two free variables (x_3 and x_4), so eigenspace $\text{Nul}(B - 2I_4)$ is **plane**.

Earlier example: for skew matrix $\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$ even though 1 appears twice on the diagonal, eigenspace with eigenvalue 1 is line x_1 axis.

In all the examples so far we are given (or can easily find) an eigenvalue.

How do you **find** an eigenvalue λ for **non-triangular** matrices?

Recall: Let B be an $n \times n$ matrix. Then

$$\text{Nul } B = \vec{0} \iff B \text{ invertible} \iff \det B \neq 0.$$

Translation: let $\lambda \in \mathbb{R}$ and $B = A - \lambda I_n$.

$$\lambda \text{ is eigenvalue for } A \iff \text{Nul } B \neq \vec{0} \iff \det B = 0.$$

In other words, the **eigenvalues** of A are exactly the values λ for which **$\det(A - \lambda I_n) = 0$** !

Since $\det(A - \lambda I_n)$ is a polynomial, this is a familiar problem.

iClicker Question

For the matrix $A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$ what is $\det(A - \lambda I_2)$?

- A) 4
- B) $4 - \lambda^2$
- C) $4 + \lambda^2$
- D) $4 - 5\lambda + \lambda^2$
- E) I have no idea.

Answer:

$$\det \begin{bmatrix} 3 - \lambda & 1 \\ 2 & 2 - \lambda \end{bmatrix} = (3 - \lambda)(2 - \lambda) - 2 = 6 - 5\lambda + \lambda^2 - 2 =$$
$$4 - 5\lambda + \lambda^2$$

Usually written $\lambda^2 - 5\lambda + 4$, a quadratic function of λ .

Example: Find all eigenvalues for the matrix $A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$.

Answer: Find values of λ for which

$$\det(A - \lambda I_2) = \det\left(\begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\begin{bmatrix} 3 - \lambda & 1 \\ 2 & 2 - \lambda \end{bmatrix} = 0$$

In other words, find roots of the quadratic equation

$$\lambda^2 - 5\lambda + 4 = 0.$$

We've known since high school how to do this!

$$\lambda^2 - 5\lambda + 4 = (\lambda - 4)(\lambda - 1) = 0 \implies \lambda = 1 \text{ or } 4.$$

Check: Both $A - 4I = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}$ and $A - I = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$ have obvious null-spaces.

Example: Find all eigenvalues and corresponding eigenvectors for the matrix $A = \begin{bmatrix} 5 & 6 \\ 3 & -2 \end{bmatrix}$.

Solution:

Step 1: Find the eigenvalues. That is, find all the solutions to the **characteristic equation** for A :

$$\det(A - \lambda I) = \det \begin{bmatrix} 5 - \lambda & 6 \\ 3 & -2 - \lambda \end{bmatrix} = 0$$

Calculate:

$$\det \begin{bmatrix} 5 - \lambda & 6 \\ 3 & -2 - \lambda \end{bmatrix} = (5 - \lambda)(-2 - \lambda) - 18 = \lambda^2 - 3\lambda - 28$$

(Note $\lambda^2 - 3\lambda - 28$ is called the **characteristic polynomial** of A .)

$$\lambda^2 - 3\lambda - 28 = (\lambda - 7)(\lambda + 4) = 0 \iff \lambda = 7 \text{ or } \lambda = -4$$

Hence eigenvalues are $\lambda = 7$ and $\lambda = -4$.

Step 2: Calculate eigenvectors for eigenvalue $\lambda = 7$:

This is the kernel (nullspace) of

$$A - 7 \cdot I_2 = \begin{bmatrix} 5-7 & 6 \\ 3 & -2-7 \end{bmatrix} = \begin{bmatrix} -2 & 6 \\ 3 & -9 \end{bmatrix} \xrightarrow{\text{reduce}} \begin{bmatrix} -2 & 6 \\ 0 & 0 \end{bmatrix}$$

Solution: Eigenspace is all multiples of vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Check:

$$\begin{bmatrix} 5 & 6 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 21 \\ 7 \end{bmatrix} = 7 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$



Step 3: Calculate eigenvectors for eigenvalue $\lambda = -4$:
This is the kernel (nullspace) of

$$A - (-4) \cdot I_2 = \begin{bmatrix} 5+4 & 6 \\ 3 & -2+4 \end{bmatrix} = \begin{bmatrix} 9 & 6 \\ 3 & 2 \end{bmatrix} \xrightarrow{\text{reduce}} \begin{bmatrix} 9 & 6 \\ 0 & 0 \end{bmatrix}$$

Solution: Eigenspace is all multiples of vector $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$

Check:

$$\begin{bmatrix} 5 & 6 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -8 \\ 12 \end{bmatrix} = -4 \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$



iClicker Question

Find the eigenvalues for the matrix $A = \begin{bmatrix} 3/2 & -13 \\ 1/4 & -3/2 \end{bmatrix}$.

- A) There are no eigenvalues.
- B) $\lambda = \pm 1$
- C) $\lambda = \pm 1/2$
- D) I have no idea.

Discussion: $\det A - \lambda I_2 = \lambda^2 + 1$.

There are no (real) solutions to $\lambda^2 + 1 = 0$.

Hence there are **no (real)** eigenvalues, hence no eigenvectors.

Alternate answer:

We saw on the midterm that $A = -A^{-1}$ or $A^2 = -I_2$.

But if $\vec{v}$ is eigenvector for A and λ its eigenvalue then $A\vec{v} = \lambda\vec{v}$.

This means $-\vec{v} = -I_2\vec{v} = A^2\vec{v} = A(\lambda\vec{v}) = \lambda^2\vec{v} \implies \lambda^2 = -1$.

Why should we care about eigenvalues and eigenvectors?

One reason:

Suppose you plan to apply a matrix linear transformation A to a vector $\vec{v}$ lots of times: $AAAAA...A\vec{v} = A^{\text{bazillion}}\vec{v}$.

If $\vec{v}$ is an eigenvector and λ is its eigenvalue, then $A^{\text{bazillion}}\vec{v} = \lambda^{\text{bazillion}}\vec{v}$. So it's easy to calculate.

Example (from above): Let $A = \begin{bmatrix} 5 & 6 \\ 3 & -2 \end{bmatrix}$. What is $A^{10} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$?

Solution: Since $A \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 7 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$,

$$AAAAAAAAAA \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 7 \cdot 7 \cdot \dots \cdot 7 \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \cdot 7^{10} \\ 7^{10} \end{bmatrix}$$

Pushing further - can often get complete formula for $A^{\text{bazillion}}$.

Example: Have shown $A \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 7 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $A \begin{bmatrix} 2 \\ -3 \end{bmatrix} = -4 \begin{bmatrix} 2 \\ -3 \end{bmatrix}$.

Combining these vector equations into a matrix equation:

$$A \begin{bmatrix} 3 & 2 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 7 \cdot 3 & -4 \cdot 2 \\ 7 \cdot 1 & -4 \cdot -3 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 7 & 0 \\ 0 & -4 \end{bmatrix}$$

If we denote $\begin{bmatrix} 3 & 2 \\ 1 & -3 \end{bmatrix} = P$ we have

$$AP = P \begin{bmatrix} 7 & 0 \\ 0 & -4 \end{bmatrix} \implies A = P \begin{bmatrix} 7 & 0 \\ 0 & -4 \end{bmatrix} P^{-1} \implies$$

$$A^{1000} = P \begin{bmatrix} 7^{1000} & 0 \\ 0 & (-4)^{1000} \end{bmatrix} P^{-1}$$

Why ever need to calculate A^{1000} ?

Example Suppose w_0 is the number of wolves in the forest, r_0 is the number of rabbits. Consider what happens:

- Wolves make more wolves (just like birds make more birds and bees make more bees). But only if the wolves eat rabbits.
- Similarly rabbits make (lots) more rabbits. But rabbits also get eaten by wolves.

So, if in year zero there are w_0 wolves & r_0 rabbits, then **next year**, the number of wolves w_1 and rabbits r_1 might be given by:

$$w_1 = \frac{w_0}{2} + \frac{2r_0}{5} \quad (1)$$

$$r_1 = -\frac{w_0}{10} + \frac{3r_0}{2} \quad (2)$$

In other words,
$$\begin{bmatrix} w_1 \\ r_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{2}{5} \\ -\frac{1}{10} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} w_0 \\ r_0 \end{bmatrix}.$$

If the same model applies the following year, then after that second year there are w_2 wolves & r_2 rabbits where

$$\begin{bmatrix} w_2 \\ r_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{2}{5} \\ -\frac{1}{10} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} w_1 \\ r_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{2}{5} \\ -\frac{1}{10} & \frac{3}{2} \end{bmatrix} \left(\begin{bmatrix} \frac{1}{2} & \frac{2}{5} \\ -\frac{1}{10} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} w_0 \\ r_0 \end{bmatrix} \right)$$

In other words,

$$\text{for } A = \begin{bmatrix} \frac{1}{2} & \frac{2}{5} \\ -\frac{1}{10} & \frac{3}{2} \end{bmatrix}, \quad \begin{bmatrix} w_2 \\ r_2 \end{bmatrix} = A^2 \begin{bmatrix} w_0 \\ r_0 \end{bmatrix}$$

More generally, after k seasons have w_k wolves & r_k rabbits where :

$$\begin{bmatrix} w_k \\ r_k \end{bmatrix} = A^k \begin{bmatrix} w_0 \\ r_0 \end{bmatrix}$$

Characteristic equation of $\begin{bmatrix} \frac{1}{2} & \frac{2}{5} \\ -\frac{1}{10} & \frac{3}{2} \end{bmatrix}$ is $\lambda^2 - 2\lambda + \frac{79}{100}$.

Roots of $\lambda^2 - 2\lambda + \frac{79}{100} = 0$ can be found via the quadratic formula:

$$\lambda = \frac{2 \pm \sqrt{4 - \frac{79}{25}}}{2} = 1 \pm \sqrt{\frac{21}{100}}$$

Now find eigenvectors for each, and assemble into the matrix P .
Then

$$A^{1000} = P \begin{bmatrix} (1 + \sqrt{\frac{21}{100}})^{1000} & 0 \\ 0 & (1 - \sqrt{\frac{21}{100}})^{1000} \end{bmatrix} P^{-1}$$

Wolves and rabbits after 1000 years!

Final complication: If A is $n \times n$ then the characteristic equation

$$\det(A - \lambda I_n) = 0$$

is a **complicated** (think cofactor expansion) degree n polynomial.

What's more: there is no general solution (like the quadratic formula) to degree n polynomials once $n \geq 5$.

And there never will be...

This is a great story with an interesting history - take Math 111 to find out more.

In practice, there are very good algorithms for **estimating** the roots.