

# Diagonalization

Math 4A – Scharlemann

4 March 2015



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**Example from last time:** In year zero there are  $w_0$  wolves &  $r_0$  rabbits, the **next year**, the number of wolves  $w_1$  and rabbits  $r_1$  might be given by:

$$\begin{bmatrix} w_1 \\ r_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{2}{5} \\ -\frac{1}{10} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} w_0 \\ r_0 \end{bmatrix}$$

Call the matrix  $A = \begin{bmatrix} \frac{1}{2} & \frac{2}{5} \\ -\frac{1}{10} & \frac{3}{2} \end{bmatrix}$ . Then (assuming nothing else changes) after  $k$  seasons have  $w_k$  wolves &  $r_k$  rabbits where :

$$\begin{bmatrix} w_k \\ r_k \end{bmatrix} = A^k \begin{bmatrix} w_0 \\ r_0 \end{bmatrix}$$

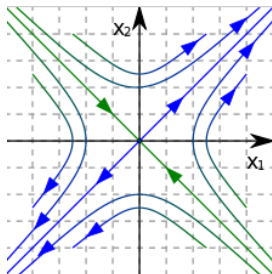
Eigenvalues for  $A$  are

$$\lambda = \frac{2 \pm \sqrt{4 - \frac{79}{25}}}{2} = 1 \pm \sqrt{\frac{21}{100}}$$

Let  $\vec{p}_1, \vec{p}_2$  be the eigenvectors for the two eigenvalues and let  $P = [\vec{p}_1 \quad \vec{p}_2]$ . Then

$$A^{1000} = P \begin{bmatrix} (1 + \sqrt{\frac{21}{100}})^{1000} & 0 \\ 0 & (1 - \sqrt{\frac{21}{100}})^{1000} \end{bmatrix} P^{-1}$$

In theory, this gives wolves and rabbits after 1000 years (assuming nothing else changes) but the geometric picture is also interesting:



In long run, “balanced ratio”: eigenvector with eigenvalue  $> 1$ .

Review from last time:

Suppose  $A$  is  $2 \times 2$  matrix  $\begin{bmatrix} 5 & 6 \\ 3 & -2 \end{bmatrix}$ .

Eigenvalues are the two roots of the characteristic equation:

$$\det(A - \lambda I) = \det \begin{bmatrix} 5 - \lambda & 6 \\ 3 & -2 - \lambda \end{bmatrix} = \lambda^2 - 3\lambda - 28 = (\lambda - 7)(\lambda + 4) = 0$$

Hence eigenvalues are  $\lambda = 7$  and  $\lambda = -4$ .

Eigenvectors for eigenvalue  $\lambda = 7$  is the kernel (nullspace) of

$$A - 7 \cdot I_n = \begin{bmatrix} -2 & 6 \\ 3 & -9 \end{bmatrix} \text{ with basis } \vec{p}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Similarly eigenspace for eigenvalue  $\lambda = -4$  has basis  $\vec{p}_2 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

Then

$$A = P \begin{bmatrix} 7 & 0 \\ 0 & -4 \end{bmatrix} P^{-1} \quad \text{where} \quad P = [\vec{p}_1 \quad \vec{p}_2] = \begin{bmatrix} 3 & 2 \\ 1 & -3 \end{bmatrix}$$

$\begin{bmatrix} 7 & 0 \\ 0 & -4 \end{bmatrix}$  is a diagonal matrix; say that we have **diagonalized**  $A$ .

### Definition

If  $A$  is a square matrix and there is an invertible matrix  $P$  so that  $A = PBP^{-1}$  then  $A$  and  $B$  are **similar**.

### Definition

If a square matrix  $A$  is similar to a diagonal matrix  $D$  then  $A$  can be **diagonalized**. The diagonalization of  $A$  is  $D$ .

Advantage of diagonalization: Easy to calculate powers of a diagonal matrix:

$$\begin{bmatrix} 7 & 0 \\ 0 & -4 \end{bmatrix}^k = \begin{bmatrix} 7^k & 0 \\ 0 & (-4)^k \end{bmatrix}$$

So if  $A$  can be diagonalized, it's easy to calculate powers of  $A$ :

$$\begin{aligned} A^k &= \left( P \begin{bmatrix} 7 & 0 \\ 0 & -4 \end{bmatrix} P^{-1} \right)^k = \\ &= \left( P \begin{bmatrix} 7 & 0 \\ 0 & -4 \end{bmatrix} P^{-1} \right) \left( P \begin{bmatrix} 7 & 0 \\ 0 & -4 \end{bmatrix} P^{-1} \right) \cdots \left( P \begin{bmatrix} 7 & 0 \\ 0 & -4 \end{bmatrix} P^{-1} \right) = \\ &= P \begin{bmatrix} 7 & 0 \\ 0 & -4 \end{bmatrix} \underbrace{(P^{-1}P)} \begin{bmatrix} 7 & 0 \\ 0 & -4 \end{bmatrix} \underbrace{(P^{-1}P)} \cdots \underbrace{(P^{-1}P)} \begin{bmatrix} 7 & 0 \\ 0 & -4 \end{bmatrix} P^{-1} = \\ &= P \left( \begin{bmatrix} 7 & 0 \\ 0 & -4 \end{bmatrix} \right)^k P^{-1} = P \begin{bmatrix} 7^k & 0 \\ 0 & (-4)^k \end{bmatrix} P^{-1} \end{aligned}$$

## iClicker Question

Suppose  $A = \begin{bmatrix} 1 & -6 \\ 2 & -6 \end{bmatrix}$  can be diagonalized. What are the diagonal entries?

- A) 1 and  $-6$
- B)  $-6$  and  $-12$
- C)  $-2$  and  $-3$
- D) I can answer this, but not in a minute.
- E) I have no idea.

Hint: characteristic polynomial is  $\lambda^2 + 5\lambda + 6$ .

## Discussion

There is a formula for the characteristic polynomial of  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ .

$$\det(A - \lambda I_2) = \det \begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix} = (\lambda^2 - d\lambda - a\lambda + ad) - bc = \lambda^2 - (a + d)\lambda + (ad - bc) = \lambda^2 - (a + d)\lambda + \det A.$$

The sum of the diagonal elements of any square matrix is called its **trace**; hence  $\text{trace } A = a + d$

So the general formula for a  $2 \times 2$  matrix  $A$  is

$$\det(A - \lambda I_2) = \lambda^2 - (\text{trace } A)\lambda + \det A$$

For the iclicker question that means

$$\lambda^2 + 5\lambda + 6 = (\lambda + 2)(\lambda + 3).$$

So the matrix has eigenvalues  $-2$  and  $-3$ . These must be the entries in the diagonalized matrix.

**Question:** When is a square matrix  $A$  similar to a diagonal matrix? That is, when can you **diagonalize**  $A$ ?

### Theorem

**Answer:** Can diagonalize  $A$  if and only if  $A$  has  $n$  linearly independent eigenvectors.

**Example:** We have seen that  $\begin{bmatrix} 5 & 6 \\ 3 & -2 \end{bmatrix}$  has eigenvector  $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$  with eigenvalue 7 and eigenvector  $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$  with eigenvalue  $-4$ .

The eigen**vectors** became columns of  $P$ ; eigen**values** became entries in  $D$ . Matrix  $P$  is invertible if and only if the eigenvectors are linearly independent.

One way to show eigenvectors are linearly independent:

Suppose  $\vec{v}_1, \dots, \vec{v}_k$  are eigenvectors, but **no two eigenvalues are the same**. Then the eigenvectors are linearly independent.

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## iClicker Question

Can  $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$  be diagonalized? What are the diagonal entries?

- A) It already is diagonal.
- B) Yes it can be diagonalized with entries 1 and 4 and 6
- C) This is a shear so it is not diagonalizable.
- D) I can answer this, but not in a minute.
- E) I have no idea.

**Example:** Let  $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ . This is a **triangular** matrix; we'll see these can be especially easy to diagonalize.

**Step 1:** Solve the characteristic equation to find the eigenvalues:  
Characteristic equation:

$$0 = \det(A - \lambda I) = \det \left( \begin{bmatrix} 1 - \lambda & 2 & 3 \\ 0 & 4 - \lambda & 5 \\ 0 & 0 & 6 - \lambda \end{bmatrix} \right) = (1 - \lambda)(4 - \lambda)(6 - \lambda)$$

**Solutions:**  $\lambda = 1, 4, 6$  are the eigenvalues.

**Moral:** for a triangular matrix, eigenvalues are the diagonal entries.  
If the diagonal entries are all different, then there are  $n$  linearly independent eigenvectors, so matrix is diagonalizable.

**Step 2:** Find eigenspace for each eigenvalue: Eigenspace for

eigenvalue **1** is the null space of  $A - I = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & 5 \end{bmatrix}$

One free variable: eigenspace spanned by single eigenvector  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Similarly, eigenspace for eigenvalues **4, 6** are each the null space of

$$A - 4I = \begin{bmatrix} -3 & 2 & 3 \\ 0 & 0 & 5 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \text{ spans eigenspace}$$

$$A - 6I = \begin{bmatrix} -5 & 2 & 3 \\ 0 & -2 & 5 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} -5 & 0 & 8 \\ 0 & -2 & 5 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 8 \\ 5 \\ 2 \\ 1 \end{bmatrix} \text{ spans eigenspace}$$

Put the eigenvectors together as columns:  $P = \begin{bmatrix} 1 & 2 & \frac{8}{5} \\ 0 & 3 & \frac{5}{2} \\ 0 & 0 & 1 \end{bmatrix}$ .

Since the columns are eigenvectors,  $AP = \begin{bmatrix} 1 \cdot 1 & 4 \cdot 2 & 6 \cdot \frac{8}{5} \\ 1 \cdot 0 & 4 \cdot 3 & 6 \cdot \frac{5}{2} \\ 1 \cdot 0 & 4 \cdot 0 & 6 \cdot 1 \end{bmatrix}$ .

This means that

$$AP = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix} \implies A = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix} P^{-1}$$

Sample application:

$$A^{52} = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4^{52} & 0 \\ 0 & 0 & 6^{52} \end{bmatrix} P^{-1}$$

**Review:** Recall how we can find  $P^{-1}$ :

**Step 1:** Augment  $P$  with the matrix  $I_3$ :

$$\left[ \begin{array}{ccc|ccc} 1 & 2 & \frac{8}{5} & 1 & 0 & 0 \\ 0 & 3 & \frac{5}{2} & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$$

Reduce to reduced echelon form:

$$\left[ \begin{array}{ccc|ccc} 1 & 2 & \frac{8}{5} & 1 & 0 & 0 \\ 0 & 3 & \frac{5}{2} & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|ccc} 1 & 2 & \frac{8}{5} & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{6} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right] \rightarrow$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{15} & 1 & -\frac{2}{3} & 0 \\ 0 & 1 & \frac{5}{6} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -\frac{2}{3} & \frac{1}{15} \\ 0 & 1 & 0 & 0 & \frac{1}{3} & -\frac{5}{6} \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$$

$$\text{So } P^{-1} = \begin{bmatrix} 1 & -\frac{2}{3} & \frac{1}{15} \\ 0 & \frac{1}{3} & -\frac{5}{6} \\ 0 & 0 & 1 \end{bmatrix}$$

Check:

$$P^{-1}P = \begin{bmatrix} 1 & -\frac{2}{3} & \frac{1}{15} \\ 0 & \frac{1}{3} & -\frac{5}{6} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 8 \\ 0 & 3 & \frac{5}{2} \\ 0 & 0 & 1 \end{bmatrix} \stackrel{??}{=} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

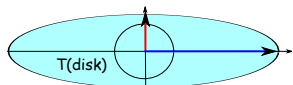
Conclusion:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 8 \\ 0 & 3 & \frac{5}{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & -\frac{2}{3} & \frac{1}{15} \\ 0 & \frac{1}{3} & -\frac{5}{6} \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow$$

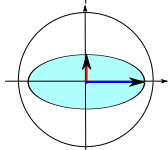
$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}^{52} = \begin{bmatrix} 1 & 2 & 8 \\ 0 & 3 & \frac{5}{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4^{52} & 0 \\ 0 & 0 & 6^{52} \end{bmatrix} \begin{bmatrix} 1 & -\frac{2}{3} & \frac{1}{15} \\ 0 & \frac{1}{3} & -\frac{5}{6} \\ 0 & 0 & 1 \end{bmatrix}$$

# Conceptual summary

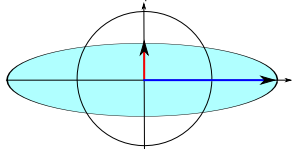
If and only if the  $n \times n$  matrix  $A$  has  $n$  linearly independent eigenvectors, there is a choice of basis (namely the eigenvectors) for which the linear transformation given by  $A$  looks very simple:



$$|Eigenvalues| > 1$$

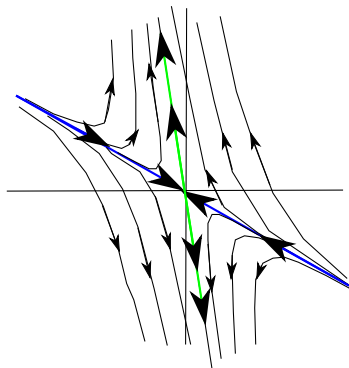


$$|Eigenvalues| < 1$$



$$|Eigenvalues| \text{ mixed}$$

But the change of basis only preserves the **topology** (general appearance) of the linear transformation not the **geometry** (distance, angles, etc.)



So let's think about geometry...(to be continued).