

Inner (dot) product

Math 4A – Scharlemann

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Suppose $\vec{u}, \vec{v} \in \mathbb{R}^n$ are given by $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

Definition

The **inner product** (also called dot product) of $\vec{u}$ and $\vec{v}$ is the real number:

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n \in \mathbb{R}$$

Sometimes it's useful to notice that this can also be written

$$\vec{u} \cdot \vec{v} = \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \vec{u}^T \vec{v}.$$

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$$\begin{bmatrix} 1 \\ -2 \\ 3 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} -8 \\ 6 \\ -4 \\ 2 \end{bmatrix} = ?$$

A) $\begin{bmatrix} -8 \\ -12 \\ -12 \\ -8 \end{bmatrix}$

B) $\begin{bmatrix} 8 \\ 12 \\ 12 \\ 8 \end{bmatrix}$

C) 40

D) -40

E) $[-8 \quad 12 \quad -12 \quad 8]$

Discussion

$$\begin{bmatrix} 1 \\ -2 \\ 3 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} -8 \\ 6 \\ -4 \\ 2 \end{bmatrix} =$$

$$1(-8) + (-2)6 + 3(-4) + (-4)2 = -8 - 12 - 12 - 8 = -40$$

Suppose we take the dot product of one of these with itself:

$$\begin{bmatrix} 1 \\ -2 \\ 3 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 3 \\ -4 \end{bmatrix} =$$

$$1 \cdot 1 + (-2) \cdot (-2) + 3 \cdot 3 + (-4) \cdot (-4) = 1^2 + 2^2 + 3^2 + 4^2 = 30$$

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$$\begin{bmatrix} -8 \\ 6 \\ -4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -8 \\ 6 \\ -4 \\ 2 \end{bmatrix} = ?$$

A) $\begin{bmatrix} 64 \\ 36 \\ 16 \\ 4 \end{bmatrix}$

B) $\begin{bmatrix} -64 \\ 36 \\ -16 \\ 4 \end{bmatrix}$

C) 120

D) -60

In general, for **any** $u \in \mathbb{R}^n$

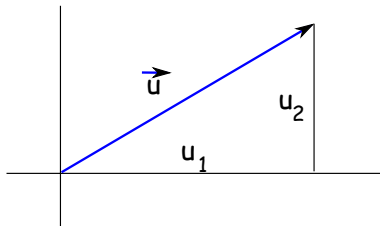
$$\vec{u} \cdot \vec{u} = u_1^2 + u_2^2 + \dots + u_n^2 \geq 0 \quad \text{and} \quad \vec{u} \cdot \vec{u} = 0 \iff \vec{u} = \vec{0}$$

Properties of inner product

For all $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and $c \in \mathbb{R}$:

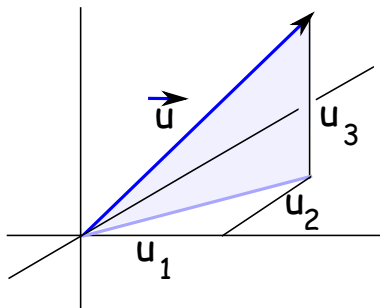
- $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
- $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$
- $(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (c\vec{v})$
- $\vec{u} \cdot \vec{u} \geq 0$, and $\vec{u} \cdot \vec{u} = 0 \iff \vec{u} = \vec{0}$.

Notice that for $\vec{u} \in \mathbb{R}^2$, $\vec{u} \cdot \vec{u} = u_1^2 + u_2^2$ is the **square of the length** of $\vec{u}$, by the **Pythagorean theorem**:



Same formula works for $\vec{u} \in \mathbb{R}^3$:

$$(\text{length of } \vec{u})^2 = (u_1^2 + u_2^2) + u_3^2 = \vec{u} \cdot \vec{u}$$



In other words

$$\text{length of } \vec{u} = \sqrt{u_1^2 + u_2^2 + u_3^2} = \sqrt{\vec{u} \cdot \vec{u}}.$$

So let's make it the general definition:

Definition

For any $\vec{u} \in \mathbb{R}^n$

- The **length** or **norm** of $\vec{u}$ is $\sqrt{\vec{u} \cdot \vec{u}}$ and is denoted $||\vec{u}||$.
- The vector $\frac{\vec{u}}{||\vec{u}||}$ is called the **unit vector** in direction of $\vec{u}$.

Explanation: What is the **length of a unit vector**? **Answer:**
Calculate

$$||\frac{\vec{u}}{||\vec{u}||}|| = \sqrt{\frac{\vec{u}}{||\vec{u}||} \cdot \frac{\vec{u}}{||\vec{u}||}} = \sqrt{\frac{\vec{u} \cdot \vec{u}}{||\vec{u}||^2}} = \sqrt{\frac{\vec{u} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}} = \sqrt{1} = 1$$

That's why it's called a **unit** vector.

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What is the unit vector in the direction of $\vec{u} = \begin{bmatrix} 1 \\ -2 \\ 3 \\ -4 \end{bmatrix}$?

A) 30

B) $\begin{bmatrix} \frac{1}{\sqrt{30}} \\ \frac{-2}{\sqrt{30}} \\ \frac{3}{\sqrt{30}} \\ \frac{-4}{\sqrt{30}} \end{bmatrix}$

C) $\begin{bmatrix} \frac{1}{30} \\ \frac{-2}{30} \\ \frac{3}{30} \\ \frac{-4}{30} \end{bmatrix}$

D) $\begin{bmatrix} 30 \\ -60 \\ 90 \\ -120 \end{bmatrix}$

E) $[-8 \quad 12 \quad -12 \quad 8]$

Discussion

What is the unit vector in the direction of $\vec{u} = \begin{bmatrix} 1 \\ -2 \\ 3 \\ -4 \end{bmatrix}$?

We have already calculated that $\vec{u} \cdot \vec{u} = 30$.
So

$$\|\vec{u}\| = \sqrt{30}$$

and the unit vector is

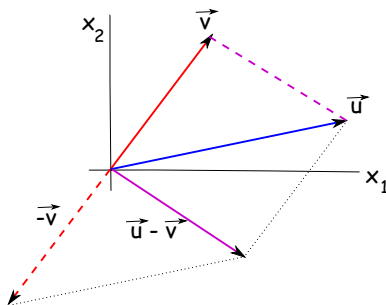
$$\frac{\vec{u}}{\|\vec{u}\|} = \frac{\vec{u}}{\sqrt{30}} = \frac{1}{\sqrt{30}} \begin{bmatrix} 1 \\ -2 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{30}} \\ \frac{-2}{\sqrt{30}} \\ \frac{3}{\sqrt{30}} \\ \frac{-4}{\sqrt{30}} \end{bmatrix}$$

Once we know the length of vectors, natural to define **distance**:

Definition

For any $\vec{u}, \vec{v} \in \mathbb{R}^n$ the **distance** between $\vec{u}$ and $\vec{v}$ is the **length** of the vector $\vec{u} - \vec{v}$. In other words:

$$\text{distance from } \vec{u} \text{ to } \vec{v} = \|\vec{u} - \vec{v}\|$$



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What's the distance between the vectors

$$\vec{u} = \begin{bmatrix} 1 \\ -2 \\ 3 \\ -4 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} -8 \\ 6 \\ -4 \\ 2 \end{bmatrix} ?$$

A) $9 + 8 + 7 + 6 = 30$

B) $7 + 4 + 1 + 2 = 14$

C) $9^2 + 8^2 + 7^2 + 6^2 = 230$

D) $7^2 + 4^2 + 1^2 + 2^2 = 70$

E) $\sqrt{230}$

Discussion:

$$\sqrt{(1 - (-8))^2 + (-2 - 6)^2 + (3 - (-4))^2 + (-4 - 2)^2} =$$

$$\sqrt{9^2 + 8^2 + 7^2 + 6^2} = \sqrt{81 + 64 + 49 + 36} = \sqrt{230}.$$

Another application of inner product:

Definition

Vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ are **orthogonal** if they are perpendicular. (This is sometimes denoted $\vec{u} \perp \vec{v}$.)

Orthogonality criterion: Amazing fact: $\vec{u} \perp \vec{v} \iff \vec{u} \cdot \vec{v} = 0$.

Example: The vectors $\vec{e}_i, \vec{e}_j$ are orthogonal $\iff i \neq j$.

Reason: If $i \neq j$ then all terms in $\vec{e}_i \cdot \vec{e}_j$ are either $0 \cdot 0$, $0 \cdot 1$ or $1 \cdot 0$.

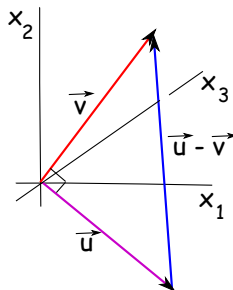
Example: In $\mathbb{R}^2$ it's easy to find perpendicular (orthogonal) vectors:

$$\text{Given } \vec{u} = \begin{bmatrix} a \\ b \end{bmatrix} \text{ let } \vec{v} = \begin{bmatrix} b \\ -a \end{bmatrix}$$

Then $\vec{u} \cdot \vec{v} = ab - ba = 0$.

$\vec{u}_\perp$ a line $\implies$ any vector perpendicular to $\vec{u}$ is a multiple of $\vec{v}$.

Why $\vec{u} \perp \vec{v} \iff \vec{u} \cdot \vec{v} = 0$?



Pythagoras says:

$$\vec{u} \perp \vec{v} \iff \|\vec{u}\|^2 + \|\vec{v}\|^2 = \|\vec{u} - \vec{v}\|^2 \iff$$

$$\vec{u} \cdot \vec{u} + \vec{v} \cdot \vec{v} = (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) = \vec{u} \cdot \vec{u} - 2\vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{v} \iff \vec{u} \cdot \vec{v} = 0$$

Weird consequence:

Problem: Suppose $\vec{u} \in \mathbb{R}^n, \vec{u} \neq \vec{0}$.

Find all vectors $\vec{v}$ so that $\vec{v} \perp \vec{u}$. This is called the **orthogonal complement** to $\vec{u}$. (Sometimes denoted $\vec{u}^\perp$).

Answer: We want a complete description of all $\vec{v}$ so that $\vec{u} \cdot \vec{v} = 0$. But remember $\vec{u} \cdot \vec{v}$ is also the matrix product $\vec{u}^T \vec{v}$.

So we are looking for the **nullspace** of the $1 \times n$ matrix $\vec{u}^T$.

In particular, the orthogonal complement $\vec{u}^\perp$ is always a **subspace** of dimension $(n - 1)$.

Problem: Find all vectors in $\mathbb{R}^3$ perpendicular to $\vec{u} = \begin{bmatrix} 1 \\ -2 \\ 3\pi \end{bmatrix}$.

It's a $3 - 1 = 2$ dimensional subspace, i. e. a plane.

Answer: It's the set of all vectors $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ so that

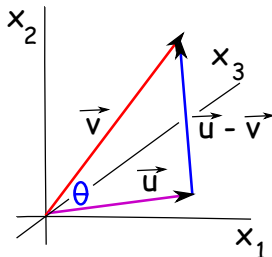
$$\vec{u} \cdot \vec{x} = 0 \quad \Longleftrightarrow \quad \vec{u}^T \vec{x} = 0 \quad \Longleftrightarrow \quad \begin{bmatrix} 1 & -2 & 3\pi \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

In other words: $1x_1 - 2x_2 + 3\pi x_3 = 0$.

You can find basis for this null-space in the usual way:

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3\pi \\ 0 \\ 1 \end{bmatrix} \right\}$$

Generalizing Pythagoras



Law of cosines says:

$$\|\vec{u} - \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos\theta \implies$$

$$\vec{u} \cdot \vec{u} + \vec{v} \cdot \vec{v} - (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) = 2\|\vec{u}\|\|\vec{v}\|\cos\theta \implies$$

$$\vec{u} \cdot \vec{u} + \vec{v} \cdot \vec{v} - (\vec{u} \cdot \vec{u} - 2\vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{v}) = 2\|\vec{u}\|\|\vec{v}\|\cos\theta \implies$$

$$2\vec{u} \cdot \vec{v} = 2\|\vec{u}\|\|\vec{v}\|\cos\theta \implies \cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{\vec{u}}{\|\vec{u}\|} \cdot \frac{\vec{v}}{\|\vec{v}\|}$$

iClicker question

We have seen that

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{||\vec{u}|| ||\vec{v}||}$$

What is the angle between the vectors $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$?

A) $\arccos \frac{1}{2}$

B) $\arccos \frac{1}{\sqrt{2}}$

C) $\arccos \frac{1}{5}$

D) $\arccos \frac{1}{\sqrt{5}}$

E) I have no idea.

Discussion $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ On the other hand

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 5 \implies \left\| \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\| = \sqrt{5}. \text{ Hence } \cos \theta = \frac{1}{\sqrt{5}}.$$