

Inner (dot) product

Math 4A – Scharlemann

8 March 2013



CELL PHONES OFF

Suppose $\vec{u}, \vec{v} \in \mathbb{R}^n$ are given by $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

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Definition

The **inner product** (also called dot product) of $\vec{u}$ and $\vec{v}$ is the real number:

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n \in \mathbb{R}$$

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Sometimes it's useful to notice that this can also be written

$$\vec{u} \cdot \vec{v} = \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \vec{u}^T \vec{v}.$$

iClicker problem

$$\begin{bmatrix} 1 \\ -2 \\ 3 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} -8 \\ 6 \\ -4 \\ 2 \end{bmatrix} = ?$$

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A) $\begin{bmatrix} -8 \\ -12 \\ -12 \\ -8 \end{bmatrix}$

B) $\begin{bmatrix} 8 \\ 12 \\ 12 \\ 8 \end{bmatrix}$

C) 40

D) -40

E) $[-8 \quad 12 \quad -12 \quad 8]$

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$$1 \cdot 1 + (-2) \cdot (-2) + 3 \cdot 3 + (-4) \cdot (-4) = 1^2 + 2^2 + 3^2 + 4^2 = 30$$

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A) $\begin{bmatrix} 64 \\ 36 \\ 16 \\ 4 \end{bmatrix}$

C) 120

B) $\begin{bmatrix} -64 \\ 36 \\ -16 \\ 4 \end{bmatrix}$

D) -60

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In general, for **any** $u \in \mathbb{R}^n$

$$\vec{u} \cdot \vec{u} = u_1^2 + u_2^2 + \dots + u_n^2 \geq 0 \quad \text{and} \quad \vec{u} \cdot \vec{u} = 0 \iff \vec{u} = \vec{0}$$

Properties of inner product

For all $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and $c \in \mathbb{R}$:

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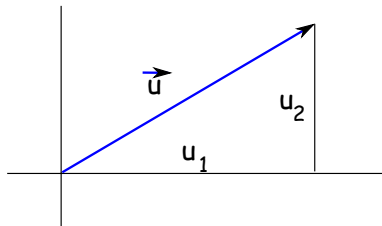
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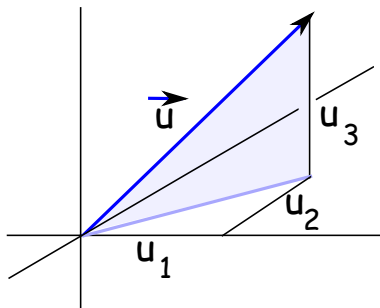
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Notice that for $\vec{u} \in \mathbb{R}^2$, $\vec{u} \cdot \vec{u} = u_1^2 + u_2^2$ is the **square of the length** of $\vec{u}$, by the **Pythagorean theorem**:



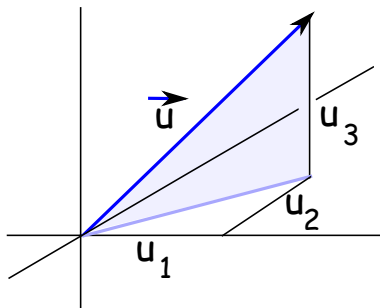
Same formula works for $\vec{u} \in \mathbb{R}^3$:

$$(\text{length of } \vec{u})^2 = (u_1^2 + u_2^2) + u_3^2 = \vec{u} \cdot \vec{u}$$



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In other words

$$\text{length of } \vec{u} = \sqrt{u_1^2 + u_2^2 + u_3^2} = \sqrt{\vec{u} \cdot \vec{u}}.$$

So let's make it the general definition:

Definition

For any $\vec{u} \in \mathbb{R}^n$

- The **length** or **norm** of $\vec{u}$ is $\sqrt{\vec{u} \cdot \vec{u}}$ and is denoted $\|\vec{u}\|$.

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Explanation: What is the **length of a unit vector**?

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Explanation: What is the **length of a unit vector**? **Answer:**
Calculate

$$||\frac{\vec{u}}{||\vec{u}||}|| = \sqrt{\frac{\vec{u}}{||\vec{u}||} \cdot \frac{\vec{u}}{||\vec{u}||}} = \sqrt{\frac{\vec{u} \cdot \vec{u}}{||\vec{u}||^2}} = \sqrt{\frac{\vec{u} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}} = \sqrt{1} = 1$$

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That's why it's called a **unit** vector.

iClicker problem

What is the unit vector in the direction of $\vec{u} = \begin{bmatrix} 1 \\ -2 \\ 3 \\ -4 \end{bmatrix}$?

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A) 30

B)

$$\begin{bmatrix} \frac{1}{\sqrt{30}} \\ \frac{-2}{\sqrt{30}} \\ \frac{3}{\sqrt{30}} \\ \frac{-4}{\sqrt{30}} \end{bmatrix}$$

C)

$$\begin{bmatrix} \frac{1}{30} \\ \frac{-2}{30} \\ \frac{3}{30} \\ \frac{-4}{30} \end{bmatrix}$$

D)

$$\begin{bmatrix} 30 \\ -60 \\ 90 \\ -120 \end{bmatrix}$$

E) $[-8 \quad 12 \quad -12 \quad 8]$

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Once we know the length of vectors, natural to define **distance**:

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For any $\vec{u}, \vec{v} \in \mathbb{R}^n$ the **distance** between $\vec{u}$ and $\vec{v}$ is the **length** of the vector $\vec{u} - \vec{v}$. In other words:

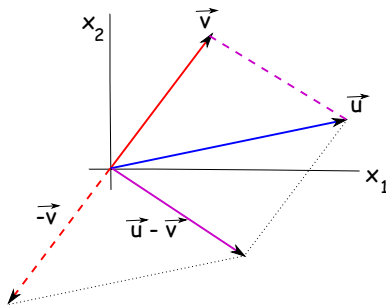
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B) $7 + 4 + 1 + 2 = 14$

C) $9^2 + 8^2 + 7^2 + 6^2 = 230$

D) $7^2 + 4^2 + 1^2 + 2^2 = 70$

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Another application of inner product:

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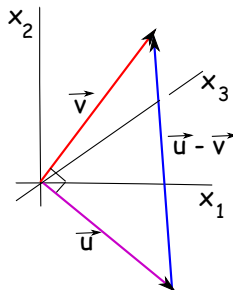
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$\vec{u} \perp$ a line $\implies$ any vector perpendicular to $\vec{u}$ is a multiple of $\vec{v}$.

Why $\vec{u} \perp \vec{v} \iff \vec{u} \cdot \vec{v} = 0$?

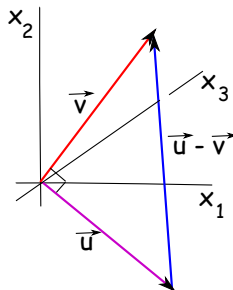
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$$\vec{u} \perp \vec{v} \iff \|\vec{u}\|^2 + \|\vec{v}\|^2 = \|\vec{u} - \vec{v}\|^2 \iff$$

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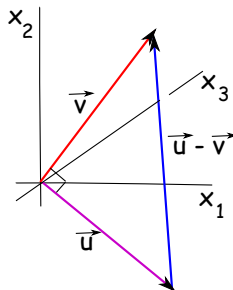


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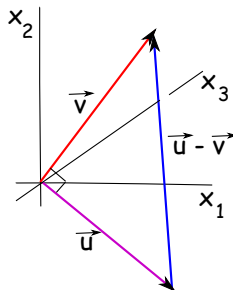


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Find all vectors $\vec{v}$ so that $\vec{v} \perp \vec{u}$. This is called the **orthogonal complement** to $\vec{u}$. (Sometimes denoted $\vec{u}^\perp$).

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In particular, the orthogonal complement $\vec{u}^\perp$ is always a **subspace** of dimension $(n - 1)$.

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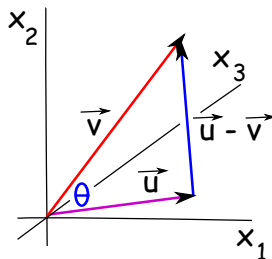
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You can find basis for this null-space in the usual way:

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3\pi \\ 0 \\ 1 \end{bmatrix} \right\}$$

Generalizing Pythagoras

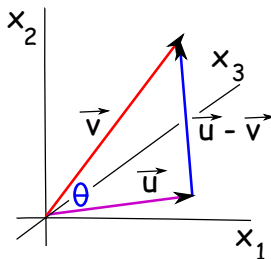
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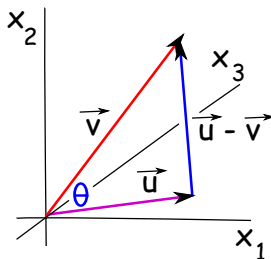


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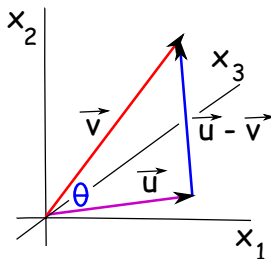
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iClicker question

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