

Getting close ('least squares')

Math 4A – Scharlemann

13 March 2013



CELL PHONES OFF

Wednesday with the Professor tonight (see syllabus)

Final: Here 8 a.m. Monday. Do only **10 (ten)** out of 12 problems.

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- Seat assignment (on Gauchospace - same as last time)
- Convincing photo id
- 8.5" by 11" bluebook for **clear written work**.

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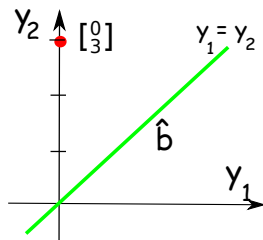
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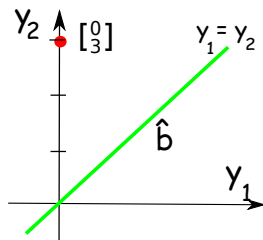
- Calculator
- Telephone or other electronics

iClicker question



Let $\hat{b}$ be the point on the line $y_1 = y_2$ at which the line gets closest to $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$, i. e. the point $y_1 = 0, y_2 = 3$. What is $\hat{b}$?

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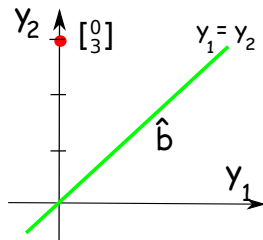
A) $\hat{b} = \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix}$

B) $\hat{b} = \begin{bmatrix} 3/2 \\ 3/2 \end{bmatrix}$

C) $\hat{b} = \begin{bmatrix} 2/3 \\ 2/3 \end{bmatrix}$

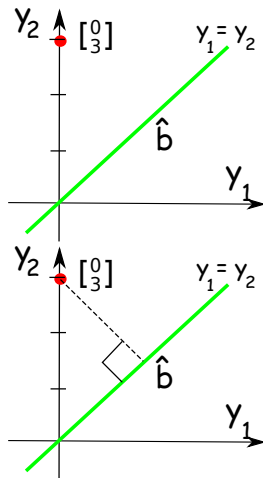
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Discussion



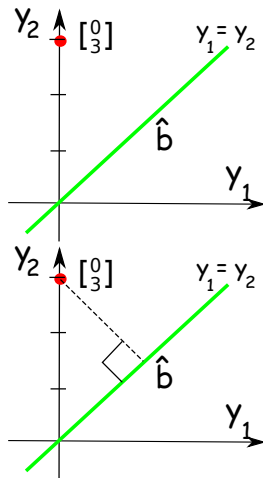
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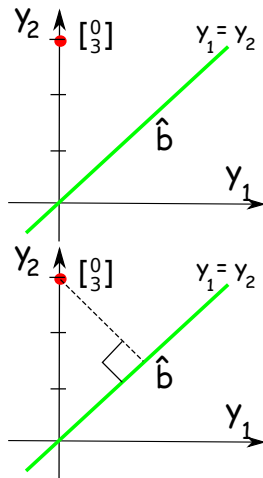
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 $\hat{b}$ is on line $y_1 = y_2$ so $= \begin{bmatrix} 3/2 \\ 3/2 \end{bmatrix}$.

Dot product verifies: $\left(\begin{bmatrix} 0 \\ 3 \end{bmatrix} - \vec{\hat{b}} \right) \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3/2 \\ 3/2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{3}{2} - \frac{3}{2} = 0$

Suppose you are given a system of linear equations that is
inconsistent:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

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And what, exactly, does this question even mean?

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But what if there is **no solution**? I.e what if the answer to the first question is “no”?

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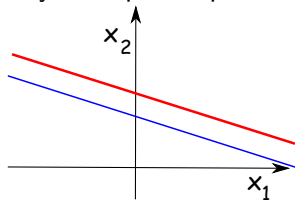
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(The hope: close enough for government work! 😊)

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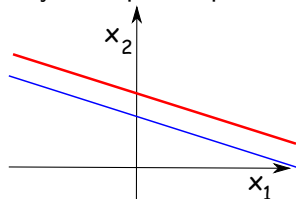


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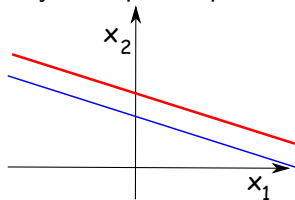
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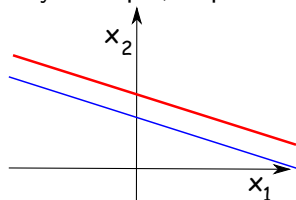
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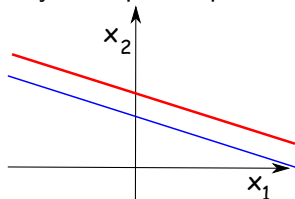
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Try to pick $\vec{x} \in \mathbb{R}^2$ so that $A\vec{x}$ is as close as possible to $\begin{bmatrix} 3 \\ 4 \end{bmatrix} \in \mathbb{R}^2$

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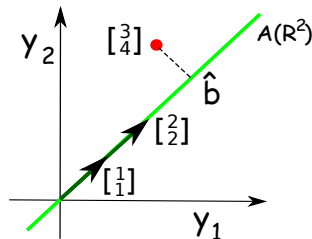
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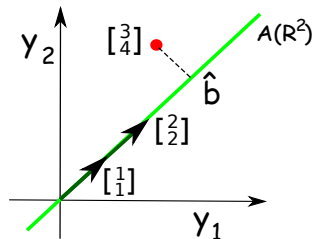
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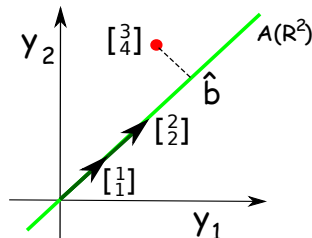
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Know two things about $\vec{b}$:

- $\vec{b} = A\vec{x}$ for **some** $\vec{x}$ and
- vector $\vec{b} - \vec{\hat{b}} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} - \vec{\hat{b}}$ is $\perp$ to $A\vec{x}$ for **all** $\vec{x}$

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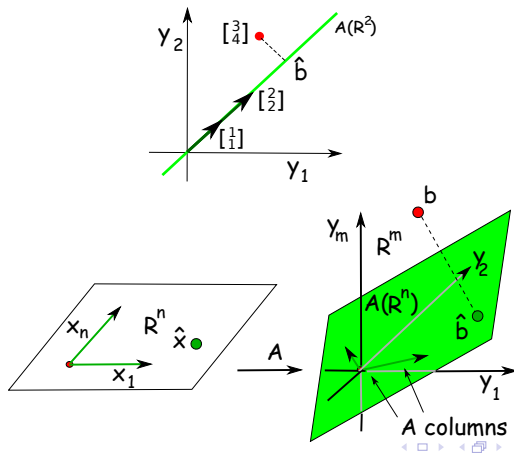
Can solve for $\vec{x}$, and then $\hat{\vec{b}} = A\vec{x}$.

General case is argued just the same – picture more complicated (more dimensions), but same outcome:

Replace $\vec{b} = A\vec{x}$ with $A^T \vec{b} = A^T A\vec{x}$.

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Example: Find the solution that gets as close as possible to

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$$-2x_1 + 4x_2 = 3$$

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$$3x_1 + 2x_2 = 1$$

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$$\begin{bmatrix} 14 & -3 \\ -3 & 21 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \end{bmatrix}$$

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Find reduced echelon form:

$$\begin{aligned} \left[\begin{array}{cc|c} 14 & -3 & 1 \\ -3 & 21 & 10 \end{array} \right] &\rightarrow \left[\begin{array}{cc|c} 14 & -3 & 1 \\ 0 & \frac{285}{14} & \frac{143}{14} \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 14 & -3 & 1 \\ 0 & 1 & \frac{143}{285} \end{array} \right] \rightarrow \\ \left[\begin{array}{cc|c} 14 & 0 & \frac{143+95}{95} \\ 0 & 1 & \frac{143}{285} \end{array} \right] &= \left[\begin{array}{cc|c} 14 & 0 & \frac{238}{95} \\ 0 & 1 & \frac{143}{285} \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & \frac{17}{95} \\ 0 & 1 & \frac{143}{285} \end{array} \right] \end{aligned}$$

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Hence

$$\begin{aligned} x_1 &= \frac{17}{95} \\ x_2 &= \frac{143}{285} \end{aligned}$$

is best (but not fully correct) solution.

Then calculate

$$\vec{\hat{b}} = A\vec{x} = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} \frac{17}{95} \\ \frac{143}{285} \end{bmatrix} = \begin{bmatrix} -\frac{92}{285} \\ \frac{439}{285} \\ \frac{94}{57} \end{bmatrix} \sim \begin{bmatrix} -0.323 \\ 1.54 \\ 1.65 \end{bmatrix}$$

Sadly, not particularly close to $\vec{b} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$.

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But we now know that it's the closest we can get!