

Vectors

Math 4A – Scharlemann

9 January 2015



CELL PHONES OFF
No office hours today

An m -vector [column vector, vector in $\mathbb{R}^m$] is an $m \times 1$ matrix:

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_m \end{bmatrix}$$

Can add two m -vectors in the obvious way, or multiply a vector by a real number:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_m \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_1 + b_2 \\ a_1 + b_3 \\ \vdots \\ a_1 + b_m \end{bmatrix} ; \quad c \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_m \end{bmatrix} = \begin{bmatrix} ca_1 \\ ca_2 \\ ca_3 \\ \vdots \\ ca_m \end{bmatrix}$$

Do not multiply two vectors together like this.

Examples:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}; \quad -1 \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$$

$$3 \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 7 \cdot \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix} + 2 \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} =$$

$$\begin{bmatrix} 3 \cdot 1 - 7 \cdot 0 + 2 \cdot 1 \\ 3 \cdot 2 - 7 \cdot 2 + 2 \cdot (-1) \\ 3 \cdot 3 - 7 \cdot 4 + 2 \cdot 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -10 \\ -19 \end{bmatrix}$$

iClicker question: Frequency AA

$$7 \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 3 \cdot \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix} - 2 \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} =$$

A) $\begin{bmatrix} 5 \\ -10 \\ -19 \end{bmatrix}$

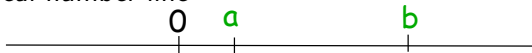
B) $\begin{bmatrix} 5 \\ -9 \\ 10 \end{bmatrix}$

C) $\begin{bmatrix} 5 \\ 10 \\ 9 \end{bmatrix}$

D) $\begin{bmatrix} 5 \\ -19 \\ -10 \end{bmatrix}$

E) $\begin{bmatrix} 5 \\ 10 \\ 19 \end{bmatrix}$

Recall the real number line



with addition (or subtraction):



(but it matters which side of 0 the numbers start out)

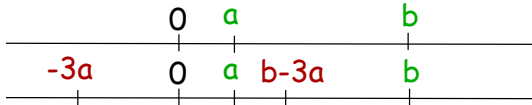


and multiplication (or division):

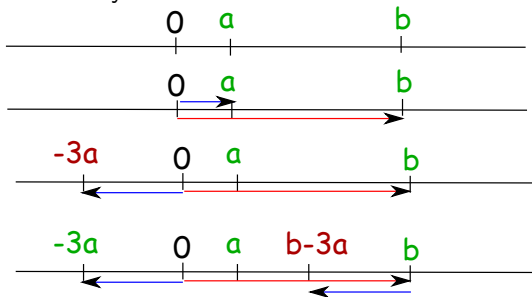


You can do this **without** knowing the actual values of a and b !

You can combine addition and multiplication:



Easier to describe if you think of numbers as arrows from 0:

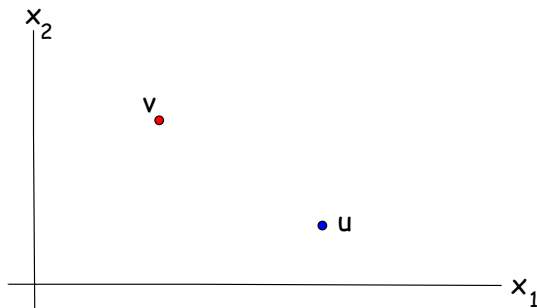


Addition: Put the head of one arrow at tail of the other.

Multiplication: Scale the length &, if negative, reverse direction

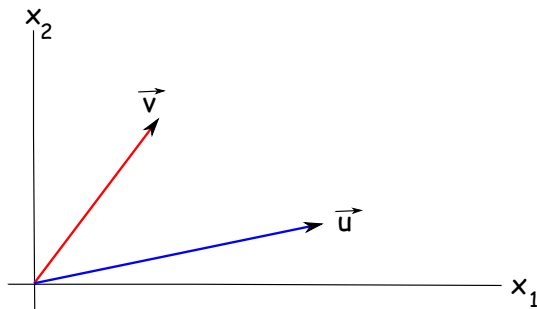
Use the same idea for 2- and 3-vectors:

- think of vector as arrow from 0
- multiply number with vector via scaling.
- add vectors head-to-tail;



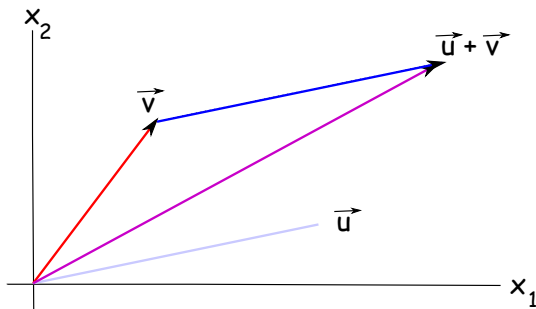
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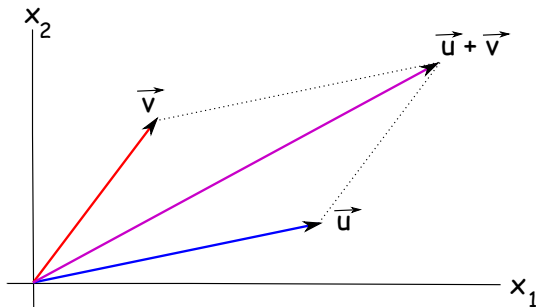
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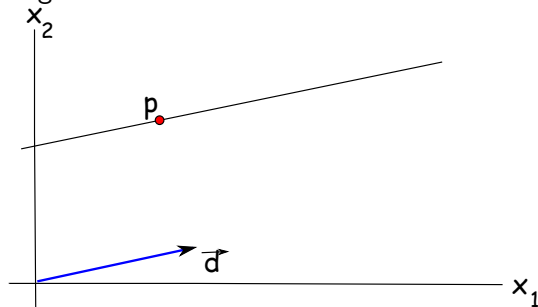
- think of vector as arrow from 0
- multiply number with vector via scaling.
- add vectors head-to-tail: **parallelogram rule**;



Application: Gives more flexible way to describe a line.
For a line through a point p , in direction $\vec{d}$, use

$$\vec{x} = \vec{p} + t \cdot \vec{d}, \quad t \in \mathbb{R}$$

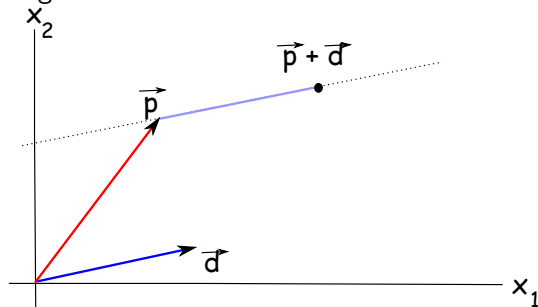
Argument:



Application: Gives more flexible way to describe a line.
For a line through a point p , in direction $\vec{d}$, use

$$\vec{x} = \vec{p} + 1 \cdot \vec{d}, \quad t = 1$$

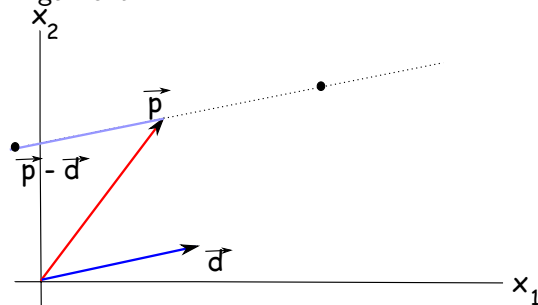
Argument:



Application: Gives more flexible way to describe a line.
For a line through a point p , in direction $\vec{d}$, use

$$\vec{x} = \vec{p} + -1 \cdot \vec{d}, \quad t = -1$$

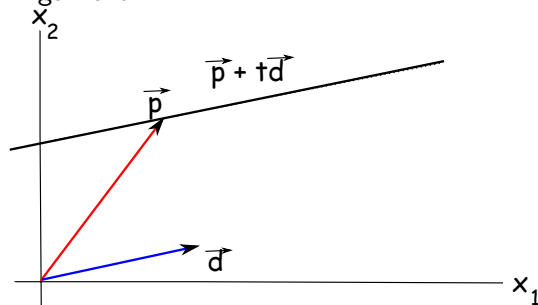
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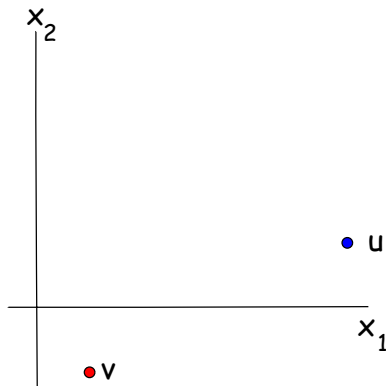
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iClicker question: Frequency AA

Pictured are points $u, v \in \mathbb{R}^2$.

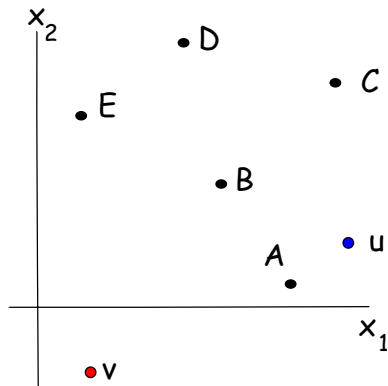
Which point represents $\vec{u} - 3\vec{v}$?



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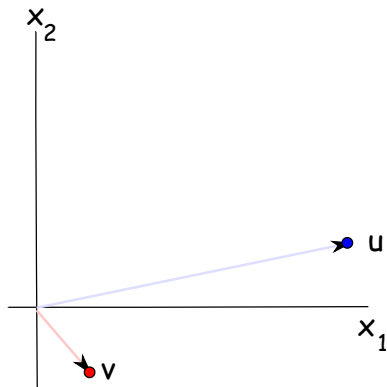
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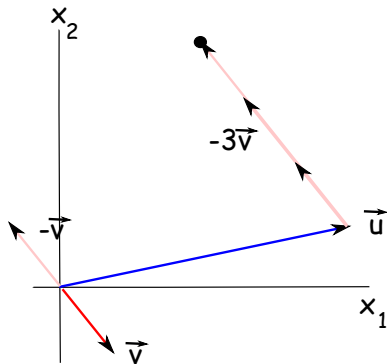
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For all $\vec{w}, \vec{x}, \vec{y} \in \mathbb{R}^n$ and $s, t \in \mathbb{R}$, we have the following.

- $\vec{x} + \vec{y} = \vec{y} + \vec{x}$ (commutative)
- $(\vec{x} + \vec{y}) + \vec{w} = \vec{x} + (\vec{y} + \vec{w})$ (associative)
- $\vec{z} + \vec{0} = \vec{0} + \vec{z} = \vec{z}$
- $\vec{x} + (-\vec{x}) = -\vec{x} + \vec{x} = \vec{0}$
- $t(\vec{x} + \vec{y}) = t\vec{x} + t\vec{y}$ (distributive law)
- $(s + t)\vec{x} = s\vec{x} + t\vec{x}$ (distributive law)
- $s(t\vec{x}) = (st)\vec{x}$ (associative)
- $1\vec{x} = \vec{x}$

Definition

Suppose $\{t_1, t_2 \dots t_k\}$ are all real numbers.

The vector

$$\vec{y} = t_1 \vec{v}_1 + \dots + t_k \vec{v}_k$$

is called a **linear combination** of the vectors $\{\vec{v}_1, \vec{v}_2 \dots \vec{v}_k\}$.

Sample problem:

Given vectors $\{\vec{a}_1, \vec{a}_2 \dots \vec{a}_n, \vec{b}\}$ in $\mathbb{R}^m$, find real numbers $\{t_1, t_2 \dots t_n\}$ so that

$$t_1 \vec{a}_1 + \dots + t_n \vec{a}_n = \vec{b}.$$

Off hand, could have any number of $\{t_1, t_2 \dots t_n\}$ solutions.

How to think about solving for $\{t_1, t_2 \dots t_n\}$ in the equation

$$t_1 \vec{a}_1 + \dots + t_n \vec{a}_n = \vec{b} :$$

Let

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix} ; \quad \vec{a}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \\ \vdots \\ a_{mj} \end{bmatrix} \quad \text{for } 1 \leq j \leq n$$

Then for any $1 \leq i \leq m$, the i^{th} row of the equation becomes:

$$t_1 a_{i1} + t_2 a_{i2} + \dots + t_n a_{in} = b_i \text{ or}$$

$$a_{i1} t_1 + a_{i2} t_2 + \dots + a_{in} t_n = b_i$$

In other words, solving

$$t_1 \vec{a}_1 + \cdots + t_n \vec{a}_n = \vec{b} :$$

is the same as solving this system of m linear equations:

$$a_{11}t_1 + \cdots + a_{1n}t_n = b_1$$

$$a_{21}t_1 + \cdots + a_{2n}t_n = b_2$$

$$\vdots$$

$$a_{m1}t_1 + \cdots + a_{mn}t_n = b_m$$

We just learned how to do this!

Solving

$$t_1 \vec{a}_1 + \cdots + t_n \vec{a}_n = \vec{b} :$$

is the same as solving a system of m linear equations.

The system has augmented matrix

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array} \right]$$

Since each $\vec{a}_j$ is a column of i numbers, can just write

$$\left[\vec{a}_1 \quad \vec{a}_2 \quad \cdots \quad \vec{a}_n \quad \vec{b} \right]$$

Example (motivated by last time):

Suppose

$$\vec{a}_1 = \begin{bmatrix} 0 \\ 2 \\ 4 \\ 8 \end{bmatrix} \quad \vec{a}_2 = \begin{bmatrix} 0 \\ 2 \\ 4 \\ 8 \end{bmatrix} \quad \vec{a}_3 = \begin{bmatrix} 6 \\ -1 \\ 1 \\ -1 \end{bmatrix} \quad \vec{a}_4 = \begin{bmatrix} 0 \\ 6 \\ 10 \\ 26 \end{bmatrix}$$

and want to find c_1, c_2, c_3, c_4 so that

$$c_1 \vec{a}_1 + c_2 \vec{a}_2 + c_3 \vec{a}_3 + c_4 \vec{a}_4 = \begin{bmatrix} 12 \\ 4 \\ 13 \\ 23 \end{bmatrix}$$

This translates to the system of linear equations whose augmented matrix is

$$\left[\begin{array}{cccc|c} 0 & 0 & 6 & 0 & 12 \\ 2 & 2 & -1 & 6 & 4 \\ 4 & 4 & 1 & 10 & 13 \\ 8 & 8 & -1 & 26 & 23 \end{array} \right]$$

which we saw last time row reduces to:

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

and so has general solution

$$c_2 = \text{anything}, \quad c_1 = \frac{3}{2} - c_2, \quad c_3 = 2, \quad c_4 = \frac{1}{2}$$

Definition

Given a collection $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ of vectors in $\mathbb{R}^m$, the set of **all linear combinations** of these vectors, that is all vectors that can be written as

$$c_1 \vec{v}_1 + \dots + c_k \vec{v}_k$$

for some $c_1, \dots, c_k \in \mathbb{R}$ is denoted

$$\text{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$$

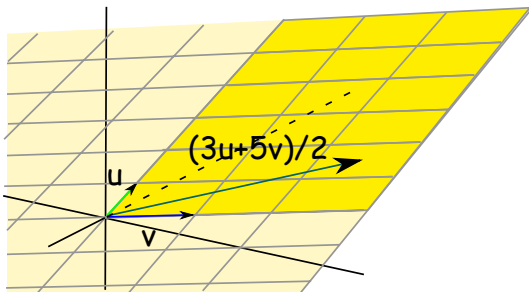
and is called the **span** of $\{\vec{v}_1, \dots, \vec{v}_k\}$.

Easy example: If $k = 1$ so there is only one vector $\vec{v}$, then $\text{Span}\{\vec{v}\}$ is just all vectors that are multiples of $\vec{v}$. That is, $\text{Span}\{\vec{v}\} = \{c\vec{v} \mid c \in \mathbb{R}\}$

Picturing the span when $m = 2, 3$:

When there is only one vector $\vec{v}$ then $\text{Span}\{\vec{v}\} = \{c\vec{v} \mid c \in \mathbb{R}\}$ is just the line that contains both $\vec{0}$ (take $c = 0$) and $\vec{v}$ (take $c = 1$).

With two vectors $\vec{u}$ and $\vec{v}$, $\text{Span}\{\vec{u}, \vec{v}\} = \{c_1\vec{u} + c_2\vec{v}\}$ pictured via the parallelogram rule (Span = entire plane; $c_i \geq 0$ highlighted):



Repeat of example above: The matrix and reduced echelon form:

$$\begin{bmatrix} 0 & 0 & 6 & 0 & 12 \\ 2 & 2 & -1 & 6 & 4 \\ 4 & 4 & 1 & 10 & 13 \\ 8 & 8 & -1 & 26 & 23 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \textcolor{brown}{1} & 0 & 0 & \vdots & \frac{3}{2} \\ 0 & \textcolor{brown}{0} & 1 & 0 & \vdots & 2 \\ 0 & \textcolor{brown}{0} & 0 & 1 & \vdots & \frac{1}{2} \\ 0 & \textcolor{brown}{0} & 0 & 0 & \vdots & 0 \end{bmatrix}$$

has general solution:

$$\textcolor{brown}{c}_2 = \textit{anything}, c_1 = \frac{3}{2} - \textcolor{brown}{c}_2, c_3 = 2, c_4 = \frac{1}{2}$$

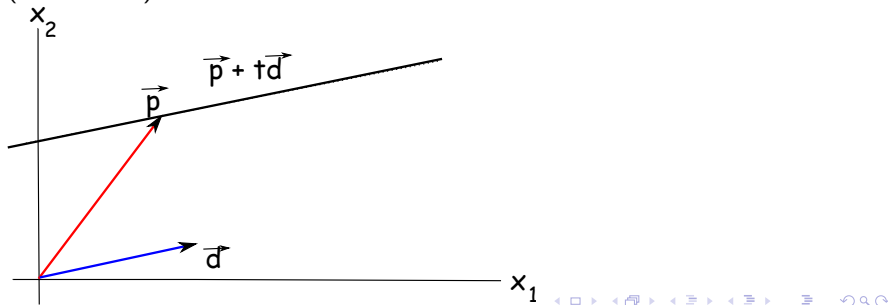
We can write this as

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 0 \\ 2 \\ \frac{1}{2} \end{bmatrix} + \textcolor{brown}{t} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

So we can think of the set of **all solutions** as

$$\begin{bmatrix} \frac{3}{2} \\ 0 \\ 2 \\ \frac{1}{2} \end{bmatrix} + \text{Span} \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

So we can picture the solution as a line in the direction of the second vector, going through the point given by the first vector (but in $\mathbb{R}^4$!)



Another from last time:

Example with two free variables $\{x_2, x_4\}$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & \pi & 0 & \frac{3}{2} \\ 0 & 0 & 1 & e & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \Rightarrow \begin{array}{l} x_1 = \frac{3}{2} - x_2 - \pi x_4 \\ x_3 = 2 - e x_4 \\ x_5 = \frac{1}{2} \end{array}$$

which can be written:

$$\begin{bmatrix} \frac{3}{2} \\ 0 \\ 2 \\ 0 \\ \frac{1}{2} \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -\pi \\ 0 \\ -e \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 0 \\ 2 \\ 0 \\ \frac{1}{2} \end{bmatrix} + \text{Span} \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\pi \\ 0 \\ -e \\ 1 \\ 0 \end{bmatrix} \right\}$$