

Matrix - Vector product

Math 4A – Scharlemann

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Recall from last time:

Definition

Suppose $\{x_1, x_2 \dots x_k\}$ are k real numbers, $\{\vec{a}_1, \vec{a}_2 \dots \vec{a}_k\}$ are k vectors.

The vector

$$x_1 \vec{a}_1 + \dots + x_k \vec{a}_k$$

is called a **linear combination** of the vectors $\{\vec{a}_1, \vec{a}_2 \dots \vec{a}_k\}$.

For example,

$$\text{Span}\{\vec{a}_1, \vec{a}_2 \dots \vec{a}_k\}$$

is the set of **all possible** linear combinations of $\{\vec{a}_1, \vec{a}_2 \dots \vec{a}_k\}$.

Think of it as a plane (which it is, if $k = 2$) in 3-space (where it is, if each $\vec{a}_i$ is in $\mathbb{R}^3$.)

Definition

The linear combination

$$x_1 \vec{a}_1 + \cdots + x_k \vec{a}_k$$

is abbreviated

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \cdots & \vec{a}_k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_k \end{bmatrix}$$

.

Here $\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \cdots & \vec{a}_k \end{bmatrix}$ is the matrix A with i^{th} column $\vec{a}_i$.

Simplest example: each $\vec{a}_i \in \mathbb{R}^1$, i. e. each column in the matrix is just a number:

$$\begin{bmatrix} a_1 & a_2 & a_3 & \dots & a_k \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_k \end{bmatrix} = a_1 b_1 + a_2 b_2 + \dots + a_k b_k.$$

so

$$\begin{bmatrix} 1 & -2 & 3 & -4 \end{bmatrix} \begin{bmatrix} 7 \\ 3 \\ 1 \\ 2 \end{bmatrix} =$$

$$1 \cdot 7 + (-2) \cdot 3 + 3 \cdot 1 + (-4) \cdot 2 = 7 - 6 + 3 - 8 = -4 \in \mathbb{R}^1.$$

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$$\begin{bmatrix} 1 & -2 & 3 & -4 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \\ 2 \\ 1 \end{bmatrix} =$$

- A) -3
- B) -6
- C) -9
- D) 6
- E) 9

More complicated example: For $\vec{a}_i \in \mathbb{R}^2, i = 1, 2, 3$:

$$\begin{bmatrix} 2 & 3 & -1 \\ 4 & -2 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 4 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ -2 \end{bmatrix} + 4 \begin{bmatrix} -1 \\ 5 \end{bmatrix}$$

so

$$\begin{bmatrix} 2 & 3 & -1 \\ 4 & -2 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \cdot 2 + 3 \cdot 1 - 1 \cdot 4 \\ 4 \cdot 2 - 2 \cdot 1 + 5 \cdot 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 26 \end{bmatrix} \in \mathbb{R}^2.$$

Think of doing the simple case on each row of the matrix A :

Apply simple case to first row:

$$\begin{bmatrix} 2 & 3 & -1 \\ 4 & -2 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \cdot 2 + 3 \cdot 1 - 1 \cdot 4 \\ 4 \cdot 2 - 2 \cdot 1 + 5 \cdot 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 26 \end{bmatrix}$$

Apply simple case to second row:

$$\begin{bmatrix} 2 & 3 & -1 \\ 4 & -2 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \cdot 2 + 3 \cdot 1 - 1 \cdot 4 \\ 4 \cdot 2 - 2 \cdot 1 + 5 \cdot 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 26 \end{bmatrix} \in \mathbb{R}^2.$$

Further abbreviation:

Use vector notation:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_k \end{bmatrix} = \vec{x}$$

then

$$x_1 \vec{a}_1 + \cdots + x_k \vec{a}_k = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \cdots & \vec{a}_k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_k \end{bmatrix} = A\vec{x}$$

Summary: For A an $m \times n$ matrix, and a vector $\vec{x} \in \mathbb{R}^n$, multiplication $A\vec{x}$ is defined and gives a vector in $\mathbb{R}^m$.

Multiplication has two important properties:

- For any vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$, $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$
- For any vector $\vec{u} \in \mathbb{R}^n$ and any $c \in \mathbb{R}$, $A(c\vec{u}) = c(A\vec{u})$.

For example:

$$A(\vec{u} + \vec{v}) = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \dots & \vec{a}_n \end{bmatrix} \left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix} \right) =$$

$$(u_1 + v_1)\vec{a}_1 + \dots + (u_n + v_n)\vec{a}_n = (u_1\vec{a}_1 + \dots + u_n\vec{a}_n) + (v_1\vec{a}_1 + \dots + v_n\vec{a}_n) = A\vec{u} + A\vec{v}$$

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$$\begin{bmatrix} 1 & -2 & 3 & -4 \\ -2 & 1 & -4 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \\ 2 \\ 1 \end{bmatrix} =$$

A) $\begin{bmatrix} 7 \\ -9 \end{bmatrix}$ B) $\begin{bmatrix} -9 \\ -4 \end{bmatrix}$ C) $\begin{bmatrix} -9 \\ 1 \end{bmatrix}$ D) $\begin{bmatrix} -4 \\ -9 \end{bmatrix}$ E) π^e

Sample problem from before:

Given vectors $\{\vec{a}_1, \vec{a}_2 \dots \vec{a}_n\}$ and $\vec{b}$ in $\mathbb{R}^m$, find real numbers $\{x_1, x_2 \dots x_n\}$ so that

$$x_1 \vec{a}_1 + \dots + x_n \vec{a}_n = \vec{b}.$$

Might arise from the question: Is $\vec{b}$ in $\text{Span}\{\vec{a}_1, \vec{a}_2 \dots \vec{a}_n\}$?

New: translation to matrix equation:

Given $m \times n$ matrix A and $\vec{b} \in \mathbb{R}^m$ find a vector $\vec{x} \in \mathbb{R}^n$ so that

$$A\vec{x} = \vec{b}$$

where $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \dots & \vec{a}_n \end{bmatrix}$. Note: the Matrix-vector multiplication $A\vec{x}$ makes sense only if the number of **columns** in A matches the number of **entries** in x .

One upshot: If A is an $m \times n$ matrix, and $\vec{b} \in \mathbb{R}^m$ then we **already know** how to find solutions (if any) to the matrix equation

$$A\vec{x} = \vec{b}:$$

- Write the augmented matrix $\left[A \mid \vec{b} \right] =$

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array} \right]$$

- Reduce to echelon form:

$$\left[\begin{array}{cccccc} \$ & * & * & * & * & * \\ 0 & \$ & * & * & * & * \\ 0 & 0 & 0 & \$ & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

- Clear out pivot columns to get reduced echelon form:

$$\begin{bmatrix} 1 & 0 & * & 0 & * & * \\ 0 & 1 & * & 0 & * & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Now solve.
- Advanced note: Solutions, if any, may be described in the form $\vec{x} \in \vec{p} + \text{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$.
Example: WebWork homework #16.

Another phrasing:

$$\vec{b} \in \text{Span}\{\vec{a}_1, \vec{a}_2 \dots \vec{a}_n\}$$

if and only if the matrix equation $A\vec{x} = \vec{b}$ has a solution.

In particular the collection of vectors $\{\vec{a}_1, \vec{a}_2 \dots \vec{a}_n\}$ span all of $\mathbb{R}^m$ if and only if for **every** $\vec{b} \in \mathbb{R}^m$ there is a vector $\vec{x} \in \mathbb{R}^n$ so that $A\vec{x} = \vec{b}$.

Example:

$$\text{Is } \begin{bmatrix} 6 \\ -5 \end{bmatrix} \text{ in } \text{Span}\left\{\begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \end{bmatrix}\right\}?$$

Translation: For

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 4 & -2 & 5 \end{bmatrix}; \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

is there a solution $\vec{x}$ to the matrix equation $A\vec{x} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$?

Answer:

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 6 \\ 4 & -2 & 5 & -5 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 2 & 3 & -1 & 6 \\ 0 & -8 & 7 & -17 \end{array} \right]$$

This is echelon form. Proceed (almost) to reduced echelon form:

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 6 \\ 0 & -8 & 7 & -17 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 2 & 0 & \frac{21-8}{8} & \frac{48-51}{8} \\ 0 & -8 & 7 & -17 \end{array} \right] =$$

$$\left[\begin{array}{ccc|c} 2 & 0 & \frac{13}{8} & \frac{-3}{8} \\ 0 & -8 & 7 & -17 \end{array} \right]$$

There is one free variable x_3 so there is a **line** of solutions.

(Note: this will be true **for any** $\vec{b}$, so the 3 vectors span **all** of $\mathbb{R}^2$.)

Let's find a solution!

$$\left[\begin{array}{ccc|c} 2 & 0 & \frac{13}{8} & \frac{-3}{8} \\ 0 & -8 & 7 & -17 \end{array} \right]$$

If set free variable $x_3 = 0$ get $2x_1 = -3/8, -8x_2 = -17$.

So $\vec{x} = \begin{bmatrix} -3/16 \\ 17/8 \\ 0 \end{bmatrix}$ should be a solution to $A\vec{x} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$.

Really, really dumb idea: don't check to see if it's right!

Let's check: $A\vec{x} =$

$$\begin{bmatrix} 2 & 3 & -1 \\ 4 & -2 & 5 \end{bmatrix} \begin{bmatrix} -3/16 \\ 17/8 \\ 0 \end{bmatrix} = \begin{bmatrix} -6/16 + 51/8 \\ -12/16 - 34/8 \end{bmatrix} =$$
$$\begin{bmatrix} 48/8 \\ -40/8 \end{bmatrix} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$$



Another solution: Set free variable $x_3 = 1$. Then

$$\left[\begin{array}{ccc|c} 2 & 0 & \frac{13}{8} & -\frac{3}{8} \\ 0 & -8 & 7 & -17 \end{array} \right]$$

gives $2x_1 + 13/8 = -3/8$ (first row) and $-8x_2 + 7 = -17 \Rightarrow$

$2x_1 = -16/8$ (first row) and $-8x_2 = -24 \Rightarrow x_1 = -1, x_2 = 3$

so $\vec{x} = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$ is another solution.

Let's check: $A\vec{x} =$

$$\begin{bmatrix} 2 & 3 & -1 \\ 4 & -2 & 5 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 + 9 - 1 \\ -4 - 6 + 5 \end{bmatrix} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$$

Exercise: General solution is

$$\vec{x} = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} + t \begin{bmatrix} -\frac{13}{16} \\ \frac{7}{8} \\ 1 \end{bmatrix} \quad t \in \mathbb{R}$$