

Solution sets

Math 4A – Scharlemann

16 January 2013



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System of linear equations is a bunch of linear equations:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

Solution to the system is a list $s_1, \dots, s_n$ of numbers so that all m equations are true when $x_i = s_i$.

There may be **lots of solutions**.

Define S to be the set of all possible solutions to the system.
Have seen that S is either empty, or a single point, or infinitely many points (with some extra structure)

New way of thinking about the problem:

A solution is a vector $\vec{x} \in \mathbb{R}^n$ that satisfies the matrix equation

$$A\vec{x} = \vec{b}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}; \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}$$

So far: S can be found by reducing the augmented matrix

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array} \right]$$

to reduced echelon form.

Then back-solve for the x_i .

When $\vec{b} = \vec{0}$ it is called a **homogeneous system** of linear equations. That is, in a homogeneous system, we are asked to find $\vec{x}$ so that

$$A\vec{x} = \vec{0}.$$

This means a system in which each $b_i = 0$, so we have

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = 0$$

A homogeneous system **always** has solution $x_1 = x_2 = \dots = x_n = 0$. This is called the **trivial solution**. The interesting question: are there **nontrivial** solutions? That is, solutions $\vec{x} \neq \vec{0}$.

Theorem

A homogeneous system has nontrivial solutions if and only if there is at least one free variable.

Reason:

If **no** free variables, then in reduced echelon form the augmented matrix is:

$$\left[\begin{array}{cccc|c} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \end{array} \right]$$

Only solution is $x_1 = x_2 = \dots = x_n = 0$.

On the other hand, if there are free variables, can set them to anything and back-substitute a solution:

For example, the reduced echelon form

$$\left[\begin{array}{ccc|c} 1 & 0 & 0.35 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

has x_3 as a free variable.

The matrix comes from the pair of equations:

$$x_1 + 0x_2 + 0.35x_3 = 0$$

$$0x_1 + 1x_2 + 0x_3 = 0$$

Forces $x_2 = 0$ but x_3 can be anything: just take $x_1 = -0.35x_3$.

For example: $x_3 = 1$, $x_1 = -0.35$ satisfies both equations.

iClicker question: Frequency AA

Joe wants to understand the solutions to the matrix equation

$$A\vec{x} = \vec{0} \quad \text{where} \quad A = \begin{bmatrix} 1 & -2 & 3 & -4 \\ 3 & 2 & -2 & 5 \\ 4 & 0 & 1 & 7 \end{bmatrix}$$

What does Joe know? (Pick the best answer)

- A) There is a solution to the equation
- B) There is a non-trivial solution to the equation
- C) There are an infinite number of solutions to the equation
- D) $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ is a non-trivial solution to the equation
- E) He should have slept in this morning.

iClicker question: Frequency AA

Joe drops the last column of the matrix, so the question becomes

$$A\vec{x} = \vec{0} \quad \text{where} \quad A = \begin{bmatrix} 1 & -2 & 3 \\ 3 & 2 & -2 \\ 4 & 0 & 1 \end{bmatrix}$$

A winged apparition, radiant in gold, appears. She points out to Joe that the last row is the **sum** of the first two rows.

What does Joe know? (Pick the best answer)

- A) There is a solution to the equation
- B) There is a non-trivial solution to the equation
- C) There are an infinite number of solutions to the equation

D) $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ is a non-trivial solution to the equation

- E) He should have played the lottery today

iClicker question: Frequency AA

Giddy with success, Joe considers

$$A\vec{x} = \vec{0} \quad \text{where} \quad A = \begin{bmatrix} 1 & 3 & 4 \\ -2 & 2 & 0 \\ 3 & -2 & 1 \end{bmatrix}$$

A horned apparition, dressed in goth, appears. He points out to Joe that the last **column** is the sum of the first two **columns**.

What does Joe know? (Pick the best answer)

- A) There is a solution to the equation
- B) There is a non-trivial solution to the equation
- C) There are an infinite number of solutions to the equation

D) $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ is a non-trivial solution to the equation

- E) He should get out more.

General fact:

If reduced echelon form has k free variables,
then there are k vectors in $\mathbb{R}^n$ which exactly span the set of solutions:

So, if exactly one free variable then S is **line** through $\vec{0}$ in $\mathbb{R}^n$.

If exactly two free variables, then S is **plane** through $\vec{0}$ in $\mathbb{R}^n$.

This gives a picture of all solutions of a homogeneous system. But
what about a **non-homogeneous** system?

That is, suppose $\vec{b} \neq \vec{0}$?

First observation:

If the **last** column of echelon form of augmented matrix is a pivot, then there is **no solution**. We say the system of equations is **inconsistent**.

$$\left[\begin{array}{ccccc|c} \$ & * & * & * & * & * \\ 0 & \$ & * & * & * & * \\ 0 & 0 & 0 & \$ & * & * \\ 0 & 0 & 0 & 0 & 0 & \$ \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$\$$ = non-zero entry (so a pivot)

The last non-zero row says $0x_1 + 0x_2 + \cdots + 0x_n = \$$. Since $\$ \neq 0$ this is impossible.

On the other hand, if the **last** column of echelon form of augmented matrix is **not a pivot**, then can back-fill **solution**. We say the system of equations is **consistent**.

To find a solution, move to reduced echelon form

$$\left[\begin{array}{cccc|c} 1 & 0 & * & 0 & * \\ 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

If **no** free variables, then solution is unique, given by the entries in the last (augmentation) column.

If there are free variables, how can we describe S ?

First observation:

If all **free** variables are set to 0 then equations easily give one solution $\vec{s}_0$:

Example: Supposed the reduced echelon form looks like

$$\left[\begin{array}{ccccc|c} 1 & 0 & * & 0 & * & \pi \\ 0 & 1 & * & 0 & * & e \\ 0 & 0 & 0 & 1 & * & \sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Set the free variables $x_3 = x_5 = 0$. Then the equations become

$$x_1 + 0x_2 + *0 + 0x_4 + *0 = \pi$$

$$0x_1 + x_2 + *0 + 0x_4 + *0 = e$$

$$0x_1 + 0x_2 + *0 + x_4 + *0 = \sqrt{2}$$

Clearly $x_1 = \pi, x_2 = e, x_4 = \sqrt{2}$ is a solution.

So far, we've found how to determine **one** solution $\vec{s}_0$. That is,

$$A\vec{s}_0 = \vec{b}$$

How to find **all** solutions?

Suppose $\vec{s}'$ is another solution, so also $A\vec{s}' = \vec{b}$.

Then

$$A(\vec{s}' - \vec{s}_0) = A\vec{s}' - A\vec{s}_0 = \vec{b} - \vec{b} = \vec{0}$$

In other words, the vector $(\vec{s}' - \vec{s}_0) = \vec{s}$ is solution to $A\vec{x} = \vec{0}$.

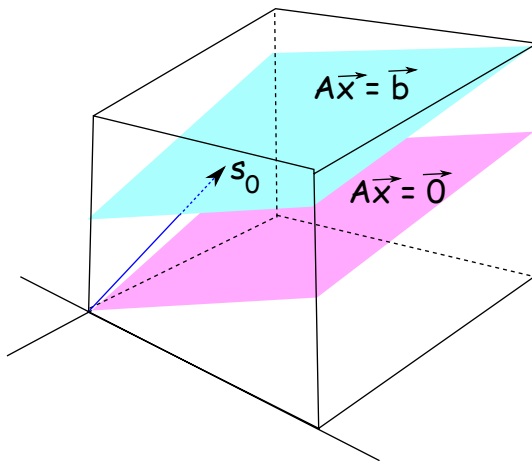
Equivalently $\vec{s}' = \vec{s}_0 + \vec{s}$.

Hence the solution set S for the system $A\vec{x} = \vec{b}$ is

$$S = \vec{s}_0 + \text{Span}\{\vec{s}_1, \dots, \vec{s}_k\}$$

where $\text{Span}\{\vec{s}_1, \dots, \vec{s}_k\}$ is solution to **homogeneous** system $A\vec{x} = \vec{0}$.

The picture in $\mathbb{R}^n$ of possible solutions to $A\vec{x} = \vec{0}$ and $A\vec{x} = \vec{b}$:



Example:
non-homogeneous system

$$x_1 + 2x_2 - 3x_3 + 2x_4 - 4x_5 = 1$$

$$2x_1 + 4x_2 - 5x_3 + 1x_4 - 6x_5 = 3$$

$$5x_1 + 10x_2 - 13x_3 + 4x_4 - 16x_5 = 7$$

Translation $A\vec{x} = \vec{b}$

homogeneous system:

$$x_1 + 2x_2 - 3x_3 + 2x_4 - 4x_5 = 0$$

$$2x_1 + 4x_2 - 5x_3 + 1x_4 - 6x_5 = 0$$

$$5x_1 + 10x_2 - 13x_3 + 4x_4 - 16x_5 = 0$$

Translation $A\vec{x} = \vec{0}$

Associated augmented matrix:

$$\left[\begin{array}{ccccc|c} 1 & 2 & -3 & 2 & -4 & 1 \\ 2 & 4 & -5 & 1 & -6 & 3 \\ 5 & 10 & -13 & 4 & -16 & 7 \end{array} \right]$$

Replace L_2 by $-2L_1 + L_2$ and L_3 by $-5L_1 + L_3$:

$$\left[\begin{array}{ccccc|c} 1 & 2 & -3 & 2 & -4 & 1 \\ 0 & 0 & 1 & -3 & 2 & 1 \\ 0 & 0 & 2 & -6 & 4 & 2 \end{array} \right]$$

Replace L_3 by $-2L_2 + L_3$:

$$\left[\begin{array}{ccccc|c} 1 & 2 & -3 & 2 & -4 & 1 \\ 0 & 0 & 1 & -3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Consistent! so there is a solution 😊 .

$$\left[\begin{array}{ccccc|c} 1 & 2 & -3 & 2 & -4 & 1 \\ 0 & 0 & 1 & -3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Replace L_1 with $L_1 + 3L_3$

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & -7 & 2 & 4 \\ 0 & 0 & 1 & -3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Free variables: x_2, x_4, x_5 . Set them to 0 and get:
 $x_1 = 4, x_3 = 1$. Hence **one** solution is

$$\vec{s}_0 = \begin{bmatrix} 4 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

For solution of the homogeneous system note

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & -7 & 2 & 0 \\ 0 & 0 & 1 & -3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

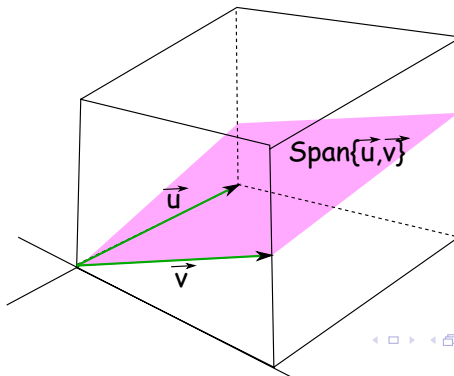
translates to

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 + 7x_4 - 2x_5 \\ x_2 \\ 3x_4 - 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 7 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

Hence **all** solutions to the **homogeneous** system given by:

$$\text{Span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Picture:



All solutions to the original **non-homogeneous** system given by:

$$\begin{bmatrix} 4 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \text{Span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Picture:

