

Applications

Math 4A – Scharlemann

16 January 2015



CELL PHONES OFF

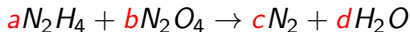
First application: **Balancing chemical Reactions**

The chemicals hydrazine (N_2H_4) and nitrogen tetroxide (N_2O_4) react to give nitrogen (N_2) and water (H_2O).

Questions: What ratio of N_2H_4 to N_2O_4 is needed to make the reaction complete?

What ratio of N_2 to H_2O is then produced?

The chemical reaction can be written:



where a, b, c, d denote the quantities of each chemical. There are as many atoms on the left as on the right, so:

$$\text{nitrogen : } 2a + 2b = 2c$$

$$\text{hydrogen : } 4a = 2d$$

$$\text{oxygen : } 4b = d$$

This translates to the homogeneous system of linear equations:

$$2a + 2b - 2c = 0$$

$$4a - 2d = 0$$

$$4b - d = 0$$

Row reduce:

$$\begin{bmatrix} 2 & 2 & -2 & 0 \\ 4 & 0 & 0 & -2 \\ 0 & 4 & 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 2 & -2 & 0 \\ 0 & -4 & 4 & -2 \\ 0 & 4 & 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & 0 & -1 \\ 0 & -4 & 4 & -2 \\ 0 & 0 & 4 & -3 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 2 & 0 & 0 & -1 \\ 0 & -4 & 0 & 1 \\ 0 & 0 & 4 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{4} \\ 0 & 0 & 1 & -\frac{3}{4} \end{bmatrix}$$

Hence, using d as the free variable:

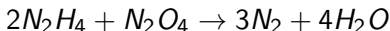
$$a = \frac{d}{2} \quad b = \frac{d}{4} \quad c = \frac{3}{4}d.$$

Hence if $\frac{a}{b} = 2$, the input chemicals will totally react.
The output will be in the ratio $\frac{c}{d} = \frac{3}{4}$.

More detail available: At the molecular level (where the solution is in integers) the smallest solution would be

$$a = 2, b = 1, c = 3, d = 4$$

Thus the best description of the chemical reaction would be:



Second application: Nutrition

CAthy is proud to be a CALifornarian from CALabasas.

To show her pride, she will only eat food that begins with CA.

Via WeightWatchers™, CAthy learns that

- In a cup of CAuliflower, 0.9 calories come from fat, 20 calories come from carbohydrates and 8 calories come from protein.
- Corresponding numbers for an ounce of CAshews are: 110, 36, 20.
- For a cup of CAntelope the numbers are 2.7, 56, 8.

iClicker question

We know that in a cup of CAuliflower, 0.9 calories come from fat, 20 calories from carbohydrates and 8 calories from protein.

Suppose CAthy eats x_1 cups of CAuliflower.

Then the vector

$$\begin{bmatrix} \text{calories from fat} \\ \text{calories from carbohydrates} \\ \text{calories from protein} \end{bmatrix} =$$

A) $\begin{bmatrix} 20 \\ 0.8 \\ 8 \end{bmatrix} x_1$

B) $\begin{bmatrix} 8 \\ 20 \\ 0.8 \end{bmatrix} x_1$

C) $\begin{bmatrix} 0.9 \\ 20 \\ 8 \end{bmatrix} x_1$

D) $\begin{bmatrix} 0.9 \\ 8 \\ 0.88 \end{bmatrix} x_1$

iClicker question

CAthy wants to eat 2000 calories.

For every calorie from fat she wants to get one calorie from protein and three from carbohydrates.

Let f be the number of calories from fat. What equation comes out of the previous sentences?

- A) $x = 2000 - f - 3f$
- B) $f + f + 3f = 2000$
- C) $x + 3f - 2000 = 0$.

What then is f ?

- A) $f = 5$
- B) $f = 4$
- C) $f = 500$
- D) $f = 400$
- E) I'm glad I'm not CAthy.

What CAthy wants:

$$\begin{bmatrix} \text{calories from fat} \\ \text{calories from carbohydrates} \\ \text{calories from protein} \end{bmatrix} = \begin{bmatrix} f \\ 3f \\ f \end{bmatrix} = \begin{bmatrix} 400 \\ 1200 \\ 400 \end{bmatrix}.$$

What WeightWatchers™ tells her:

If she eats x_1 cups of cauliflower, x_2 ounces of cashews, and x_3 cups of cantaloupe then

$$\begin{bmatrix} \text{calories from fat} \\ \text{calories from carbohydrates} \\ \text{calories from protein} \end{bmatrix} = \begin{bmatrix} 0.9 \\ 20 \\ 8 \end{bmatrix} x_1 + \begin{bmatrix} 110 \\ 36 \\ 20 \end{bmatrix} x_2 + \begin{bmatrix} 2.7 \\ 56 \\ 8 \end{bmatrix} x_3$$
$$= \begin{bmatrix} 0.9 & 110 & 2.7 \\ 20 & 36 & 56 \\ 8 & 20 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Solution: Cathy needs to solve the matrix equation:

$$\begin{bmatrix} 400 \\ 1200 \\ 400 \end{bmatrix} = \begin{bmatrix} 0.9 & 110 & 2.7 \\ 20 & 36 & 56 \\ 8 & 20 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Fortunately, she's had Math 4A! Scribble, scribble, scribble....

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 35.2 \\ 3.2 \\ 6.8 \end{bmatrix}$$

In other words, if Cathy eats 35.2 cups of Cauliflower, 3.2 ounces of Cashews, and 6.8 cups of Canteloupe, she will have achieved her dietary goal. YUMM...

Third application: Economics (Input-Output analysis)

Mr. Big runs three industries: coal, electricity, railroad.

- To mine \$1 of coal, Mr. Big needs 25¢ of electricity, 25¢ of railroad transportation.
- To generate \$1 of electricity, Mr. Big needs 65¢ of coal, 5¢ of electricity, 5¢ of railroad transportation.
- To provide \$1 of railroad transportation, Mr. Big needs 55¢ of coal, 10¢ of electricity.

iClicker question

Mr. Big gets a new order for x_1 of coal. To meet the order how much more electricity and railroad transportation will Mr. Big need to **produce**?

- A) $0.25x_1$ of electricity, and $0.25x_1$ of of railroad transportation.
- B) More than $0.25x_1$ of electricity, and $0.25x_1$ of of railroad transportation.
- C) Less than $0.25x_1$ of electricity, and $0.25x_1$ of of railroad transportation.

Discussion: To produce x_1 of coal you do need $0.25x_1$ of electricity, and $0.25x_1$ of of railroad transportation.

But to produce that electricity and the transportation you need more coal.

And that extra coal will need more electricity and more transportation, which will need yet more coal and...

This could get confusing!

Ms. Little sends Mr. Big a new order for \$50,000 of coal and \$25,000 of electricity.

How much more coal, electricity, transportation will Mr. Big need to **produce** in order to meet Ms. Little's extra **demand**?

Clarifying thought: Consider vectors whose entries are

$$\begin{bmatrix} \text{coal} \\ \text{electricity} \\ \text{transportation} \end{bmatrix}$$

Then the facts above say that the vector of goods **consumed** in

mining x_1 dollars worth of coal is $\begin{bmatrix} 0 \\ .25 \\ .25 \end{bmatrix} x_1$.

To mine x_1 dollars worth of coal consumes $\begin{bmatrix} 0 \\ .25 \\ .25 \end{bmatrix} x_1$.

To generate x_2 dollars worth of electricity consumes $\begin{bmatrix} .65 \\ .05 \\ .05 \end{bmatrix} x_2$.

To provide x_3 dollars worth of transportation consumes $\begin{bmatrix} .55 \\ .10 \\ 0 \end{bmatrix} x_3$.

In other words, to create this vector of **outputs** $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ requires this vector of **inputs**:

$$\begin{bmatrix} 0 \\ .25 \\ .25 \end{bmatrix} x_1 + \begin{bmatrix} .65 \\ .05 \\ .05 \end{bmatrix} x_2 + \begin{bmatrix} .55 \\ .10 \\ 0 \end{bmatrix} x_3 = \begin{bmatrix} 0 & .65 & .55 \\ .25 & .05 & .10 \\ .25 & .05 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The matrix

$$C = \begin{bmatrix} 0 & .65 & .55 \\ .25 & .05 & .10 \\ .25 & .05 & 0 \end{bmatrix}$$

is known as the **consumption matrix** or **input-output matrix**.

So far:

- $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ the vector of Mr. Big's outputs and
- $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ the vector of Mr. Big's inputs needed

$$\vec{y} = C\vec{x}.$$

A confusing bit: Input-Output analysis **puts in** the value $\vec{x}$ of economic **outputs** sought and, as **output** gives you the value $\vec{y}$ of **inputs** needed!

The mathematical input is the economic output and vice versa.

YIKES!

Let's go further, and figure out how much stuff Mr. Big has left over to sell to Ms. Little:

If $\vec{x}$ is produced and $\vec{y}$ is consumed in the production process, then what's left over to sell is $\vec{x} - \vec{y}$.

I. e. if the outside demand for goods is given by vector $\vec{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$,
then

$$\vec{d} = \vec{x} - \vec{y} = I_3 \vec{x} - C \vec{x} = (I_3 - C) \vec{x}.$$

What's I_3 ? A little digression...

Definition

Consider the $n \times n$ matrix with 1's on the diagonal and otherwise 0:

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}.$$

It's denoted I_n and is called the **identity matrix**.

It has this magical property: for any n -vector $\vec{x}$

$$I_n \vec{x} = \vec{x}.$$

We have made that substitution in the earlier equation to deduce

$$\vec{d} = (I_3 - C)\vec{x}.$$

(Subtraction here is done as with vectors, entry by entry.)

For Ms. Little's order, this translates to the vector-matrix equation:

$$\begin{bmatrix} 50,000 \\ 25,000 \\ 0 \end{bmatrix} = \left(\begin{bmatrix} 1 & .0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & .65 & .55 \\ .25 & .05 & .10 \\ .25 & .05 & 0 \end{bmatrix} \right) \vec{x} =$$
$$\begin{bmatrix} 1 & -.65 & -.55 \\ -.25 & .95 & -.10 \\ -.25 & -.05 & 1 \end{bmatrix} \vec{x}$$

To answer Mr. Big's question, he needs to solve

$$\begin{bmatrix} 1 & -.65 & -.55 \\ -.25 & .95 & -.10 \\ -.25 & -.05 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 50,000 \\ 25,000 \\ 0 \end{bmatrix}$$

Fortunately, Mr. Big has taken Math 4A!

$$\begin{bmatrix} 1 & -.65 & -.55 \\ -.25 & .95 & -.10 \\ -.25 & -.05 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 50,000 \\ 25,000 \\ 0 \end{bmatrix} \Rightarrow \text{scribble, scribble...}$$

$$\vec{x} = \begin{bmatrix} 102087 \\ 56163 \\ 28330 \end{bmatrix}$$

Thus Mr. Big will need to produce a total of \$102,087 of coal, \$56,163 of electricity and \$28,330 of transportation to meet Ms. Little's order.