

Linear Independence

Math 4A – Scharlemann

21 January 2015



CELL PHONES OFF

Midterm in one week!

Must bring

- TA.RD.IS code
- Seat assignment
- Convincing photo id
- 8.5" by 11" bluebook for written work. Bring some to **sell**!

May bring

- **One** 3" x 5" card
- Headlamp

Don't bring

- Calculator
- Telephone or other electronics

Background thoughts from previous lectures:

If two non-trivial vectors $\vec{x}_1, \vec{x}_2$ both lie on the same line, then $\text{Span}\{\vec{x}_1, \vec{x}_2\}$ is just that line.

On the other hand, if they *don't* lie on the same line, then $\text{Span}\{\vec{x}_1, \vec{x}_2\}$ consists of an entire plane.
(The plane determined by the heads of the vectors and $\vec{0}$.)

So the span of two vectors may be a plane, or it could be something simpler: either a line, or even just $\vec{0}$ in the case that $\vec{x}_1 = \vec{0} = \vec{x}_2$.

Similarly, if *three* non-trivial vectors $\vec{x}_1, \vec{x}_2, \vec{x}_3$ all lie on the same line, then $\text{Span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ is just that line.

If they don't all lie on the same line, but lie on the same plane, then $\text{Span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ is just that plane.

If they don't all lie in the same plane, then $\text{Span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ looks like space.

How do we put these ideas into math lingo, so we can be precise?

Definition

A set of vectors $\{\vec{v}_1, \dots, \vec{v}_k\}$ in $\mathbb{R}^m$ is **linearly independent** if and only if the only solution to the equation

$$x_1 \vec{v}_1 + \dots + x_k \vec{v}_k = \vec{0}$$

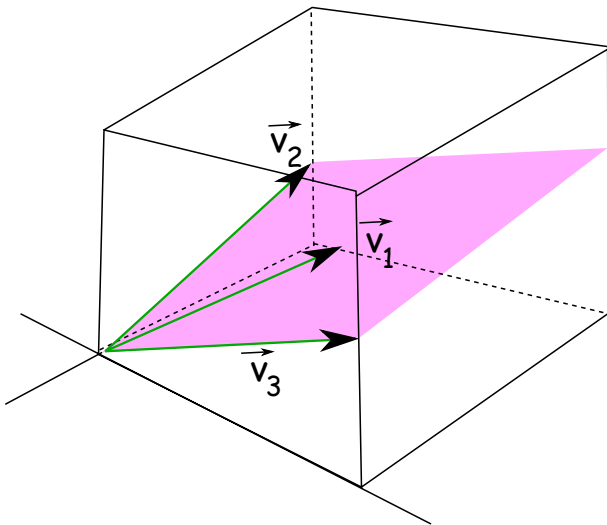
is the solution $x_i = 0$ for $1 \leq i \leq k$.

Conversely, the set of vectors $\{\vec{v}_1, \dots, \vec{v}_k\}$ is **linearly dependent** if there are real numbers $c_1, \dots, c_k$, not all zero, such that

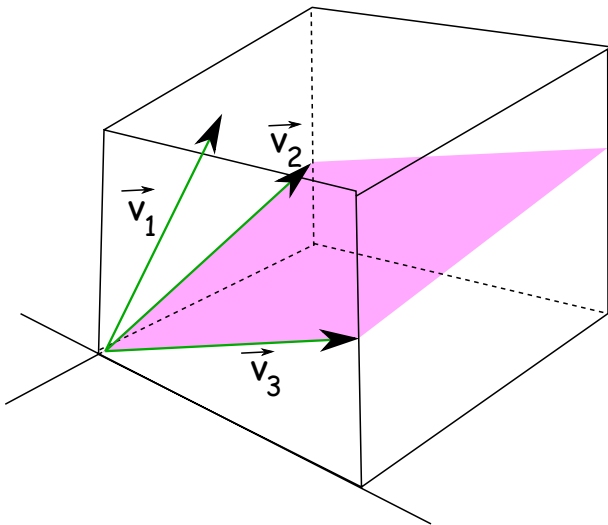
$$c_1 \vec{v}_1 + \dots + c_k \vec{v}_k = \vec{0}.$$

Idea: if the **set** is linearly **independent**, then span is big as possible. If the **set** is linearly **dependent** then span is “thinner” than it has to be; you could even throw some away and not change the span.

In pictures:



$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ linearly **dependent**.



$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ linearly independent.

Example 1: If even a single $\vec{v}_i = \vec{0}$ then $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly dependent. Why?

Suppose for concreteness that $\vec{v}_1 = \vec{0}$. Then

$$1\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3 \cdots + 0\vec{v}_k = \vec{0}$$

yet $c_1 = 1 \neq 0$.

Example 2: If even a single $\vec{v}_i$ is a multiple of a different $\vec{v}_j$ then $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly dependent. Example: Suppose for concreteness that $\vec{v}_2 = 5\vec{v}_1$. Then

$$5\vec{v}_1 - \vec{v}_2 + 0\vec{v}_3 \cdots + 0\vec{v}_k = \vec{0}$$

yet $c_1 = 5 \neq 0$ (and also $c_2 = -1 \neq 0$).

Example 3: A pair of vectors $\{\vec{u}, \vec{v}\}$ is linearly dependent if and only if at least one of the vectors is a multiple of the other.

We've just shown that if one is a multiple of the other then they are linearly dependent. - we need to show the converse: if two vectors are linearly dependent, then one is a multiple of the others.

Argument: Suppose they are linearly dependent. Then there are $c, d \in \mathbb{R}$, not both 0, so that

$$c\vec{u} + d\vec{v} = \vec{0}.$$

Suppose, for example, $c \neq 0$. Then $\vec{u} + \frac{d}{c}\vec{v} = \vec{0}$ which means

$$\vec{u} = -\frac{d}{c}\vec{v}.$$

So we see that $\vec{u}$ is a multiple of $\vec{v}$.

More generally, if any **subset** of $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly dependent, so is the whole set.

Argument: Suppose, say, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \subset \{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly dependent. This means that there are c_1, c_2, c_3 not all 0 so that

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{0}$$

But then,

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 + 0\vec{v}_4 + \dots + 0\vec{v}_n = \vec{0}$$

so the whole set is linearly dependent.

Equivalently: if $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly **independent** then so is every subset of vectors from $\{\vec{v}_1, \dots, \vec{v}_k\}$.

How to check whether a set of vectors $\{\vec{a}_1, \dots, \vec{a}_n\}$ is linearly dependent: It's equivalent to saying there is **non-zero** solution to

$$x_1 \vec{a}_1 + \dots + x_n \vec{a}_n = \vec{0}.$$

But this is a homogeneous system of linear equations - we can answer that question!

Construct associated matrix of column vectors:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Reduce to **echelon form** and see if there are any **free variables**.
There are **no free variables** if and only if **linearly independent**

Example 4: Any set of **more than m** vectors in $\mathbb{R}^m$ is linearly dependent.

For if $n > m$ then the associated matrix of column vectors

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

is **longer** than it is high.

When reduced to echelon form, there must be free variables:

$$\begin{bmatrix} \$ & * & * & * & * & * \\ 0 & \$ & * & * & * & * \\ 0 & 0 & 0 & \$ & * & * \\ 0 & 0 & 0 & 0 & \$ & * \\ 0 & 0 & 0 & 0 & 0 & \$ \end{bmatrix}$$

Here $6 > 5$ and x_3 is the free variable.

iClicker question

Question: Is this set of vectors linearly dependent, or linearly independent?

$$\left\{ \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \right\}$$

- A) Dependent since they are 2 vectors in $\mathbb{R}^3$ and $2 < 3$.
- B) Independent since they are 2 vectors in $\mathbb{R}^3$ and $2 < 3$.
- C) Dependent because one is a multiple of the other.
- D) Independent because neither is a multiple of the other.
- E) Independent because one is a multiple of the other.

Answer: It's a **pair of vectors** and neither is a multiple of the other.
Hence linearly **independent**.

iClicker question

Question: Is this set of vectors linearly dependent, or linearly independent?

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ -13 \end{bmatrix} \right\}$$

- A) Independent since they are 3 vectors in $\mathbb{R}^3$.
- B) Dependent because one is a multiple of the other.
- C) Dependent because a subset is dependent.
- D) Independent because a subset is independent.

Answer: The second is a multiple of the first, so that pair alone is linearly **dependent**. Since this subset is linearly dependent, so is the entire set. Both B) and C) are used to show linear dependence.

iClicker question

Question: Is this set of vectors linearly dependent, or linearly independent?

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix} \right\}$$

- A) Independent since they are 3 vectors in $\mathbb{R}^3$.
- B) Dependent because one is a multiple of the other.
- C) Dependent because a subset is dependent.
- D) Independent because a subset is independent.
- E) I can't tell.

Answer: For this triple of vectors

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix} \right\}$$

we want to determine whether the homogeneous system of linear equations:

$$\begin{bmatrix} 1 & 1 & 4 \\ 1 & 3 & 9 \\ 0 & 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{0}$$

has a non-trivial solution.

Consider the associated column matrix (no need to augment):

$$\begin{bmatrix} 1 & 1 & 4 \\ 1 & 3 & 9 \\ 0 & 2 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 4 \\ 0 & 2 & 5 \\ 0 & 2 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

Since there is a **free variable** (namely x_3) there are non-trivial solutions, so linearly **dependent**.

More detail on solution via **reduced** echelon form

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1.5 \\ 0 & 1 & 2.5 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1.5 \\ -2.5 \\ 1 \end{bmatrix} \text{ is one solution}$$

Check:

$$\begin{bmatrix} 1 & 1 & 4 \\ 1 & 3 & 9 \\ 0 & 2 & 5 \end{bmatrix} \begin{bmatrix} -1.5 \\ -2.5 \\ 1 \end{bmatrix} = -1.5 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - 2.5 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So these vectors linearly **dependent**.

Theorem

*A set of vectors $\{\vec{v}_1, \dots, \vec{v}_k\}$ in $\mathbb{R}^m$ is linearly **dependent** if and only if at least one of the vectors is in the span of all the others.*

For example, suppose $\vec{v}_1 \in \text{Span}\{\vec{v}_2, \vec{v}_3, \dots, \vec{v}_k\}$. That means there are $c_2, c_3, \dots, c_k$ so that

$$\vec{v}_1 = c_2 \vec{v}_2 + c_3 \vec{v}_3 + \dots + c_k \vec{v}_k$$

But this vector equation can be rewritten

$$-1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + \dots + c_k\vec{v}_k = \vec{0}.$$

Since at least one of the coefficients (namely -1) is not zero, this shows the set of vectors is linearly dependent.

Theorem

*A set of vectors $\{\vec{v}_1, \dots, \vec{v}_k\}$ in $\mathbb{R}^m$ is linearly **dependent** if and only if at least one of the vectors is in the span of all the others.*

Here's the argument in the other direction: Suppose $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly dependent. That means there are $c_1, c_2, \dots, c_k$, **not all zero**, so that

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 + \dots + c_k \vec{v}_k = \vec{0}.$$

For concreteness, say $c_1 \neq 0$. Divide by c_1 to get

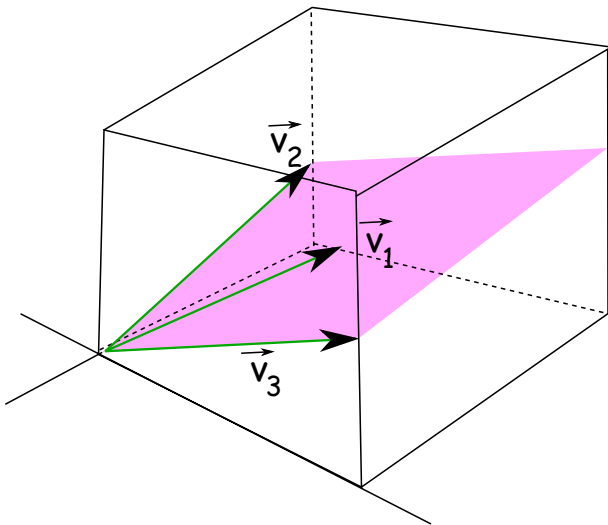
$$\vec{v}_1 + \frac{c_2}{c_1} \vec{v}_2 + \frac{c_3}{c_1} \vec{v}_3 + \dots + \frac{c_k}{c_1} \vec{v}_k = \vec{0}$$

and this can be rewritten

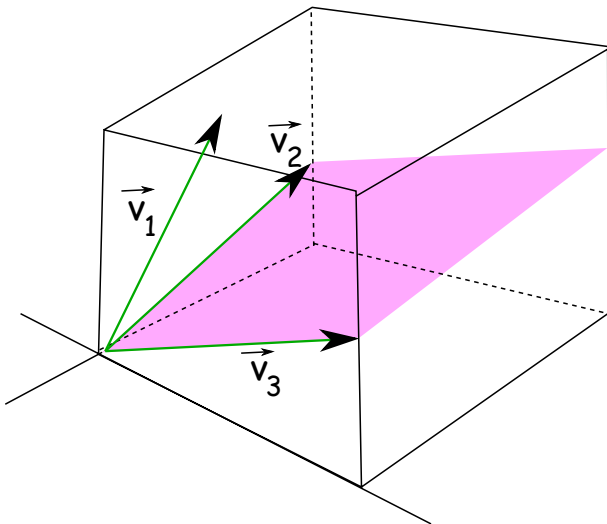
$$\vec{v}_1 = -\frac{c_2}{c_1} \vec{v}_2 - \frac{c_3}{c_1} \vec{v}_3 - \dots - \frac{c_k}{c_1} \vec{v}_k$$

Hence $\vec{v}_1 \in \text{Span}\{\vec{v}_2, \vec{v}_3, \dots, \vec{v}_k\}$.

In pictures:



$\vec{v}_1 \in \text{Span}\{\vec{v}_2, \vec{v}_3\}$ and $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ linearly **dependent**.



$\vec{v}_1 \notin \text{Span}\{\vec{v}_2, \vec{v}_3\}$ and $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ linearly independent.