

Linear Transformations

Math 4A – Scharlemann

23 January 2015



CELL PHONES OFF

Midterm Wednesday! Practice Sitdown Monday!

Must bring

- TA.RD.IS code (now a grade comment on Gauchospace)
- Seat assignment (now a grade comment on Gauchospace)
- Convincing photo id
- 8.5" by 11" bluebook for written work. Bring some to **sell**!

May bring

- **One** 3" x 5" card
- Headlamp

Don't bring

- Calculator
- Telephone or other electronics

Definition

A **function** f from a set X to a set Y is a rule that assigns to each element in X an element in Y .

Such a function is written $f : X \rightarrow Y$. The element of Y that is assigned to $x \in X$ is written $f(x)$. X is called the **domain** of f ; the subset of Y containing all points $f(x) | x \in X$ is called the **image** of f (sometimes called the range of f).

Example: Let X be the set of students in the room; Y be the set of names in the universe. There is a **naming** function

$$n : X \rightarrow Y$$

which assigns to each person in the room, his/her name. Each person has one name. (But two people may have the same name and not every name is used up.)

The functions we're interested in are **linear transformations**:

Definition

A **linear transformation** is a function $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ with these properties:

- For any vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$, $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$
- For any vector $\vec{u} \in \mathbb{R}^n$ and any $c \in \mathbb{R}$, $T(c\vec{u}) = cT(\vec{u})$.

Example: Let $T : \mathbb{R}^1 \rightarrow \mathbb{R}^1$ be defined by $T(x) = 5x$.

For any $u, v \in \mathbb{R}^1$,

- $T(u + v) = 5(u + v) = 5u + 5v = T(u) + T(v)$ and
- for any $c \in \mathbb{R}$, $T(cu) = 5cu = c5u = cT(u)$.

So T is a linear transformation.

iClicker question

Question: Let $T : \mathbb{R}^1 \rightarrow \mathbb{R}^1$ be given by $T(x) = x^2$. Is T a linear transformation?

- A) Yes.
- B) No, because sometimes $T(u + v) \neq T(u) + T(v)$.
- C) No, because always $T(u + v) \neq T(u) + T(v)$.
- D) No, because sometimes $T(cu) \neq cT(u)$.
- E) No, because always $T(cu) \neq cT(u)$.

Same question for $T : \mathbb{R}^1 \rightarrow \mathbb{R}^1$ given by $T(x) = x + 5$.

Some notes:

- A linear transformation always takes **vectors to vectors**. (Not definable on arbitrary set.)
- Most functions are **not** linear transformations. For example: $\cos(x + y) \neq \cos(x) + \cos(y)$. Or previous iclicker question.
- For **any** linear transformation $T(\vec{0}) = \vec{0}$: Take $c = 0$, then

$$T(\vec{0}) = T(0 \cdot \vec{0}) = 0T(\vec{0}) = \vec{0}.$$

- The two conditions could be written as one: For any vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ and real numbers $a, b \in \mathbb{R}$,
 $T(a\vec{u} + b\vec{v}) = aT(\vec{u}) + bT(\vec{v})$.

Important example:

Let A be any $m \times n$ matrix. Define $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ by $T(\vec{x}) = A\vec{x}$. We have already seen that T has what it takes:

- For any vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$,
$$T(\vec{u} + \vec{v}) = A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v} = T(\vec{u}) + T(\vec{v})$$
- For any vector $\vec{u} \in \mathbb{R}^n$ and any $c \in \mathbb{R}$,
$$T(c\vec{u}) = A(c\vec{u}) = c(A\vec{u}) = cT(\vec{u}).$$

A linear transformation defined by a matrix is called a **matrix transformation**

Matrix transformations are **cool**!

Example 1, a shear: Consider the matrix transformation
 $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by the 2×2 matrix

$$A = \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 1 \end{bmatrix}$$

For any **horizontal** vector $\vec{x} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$

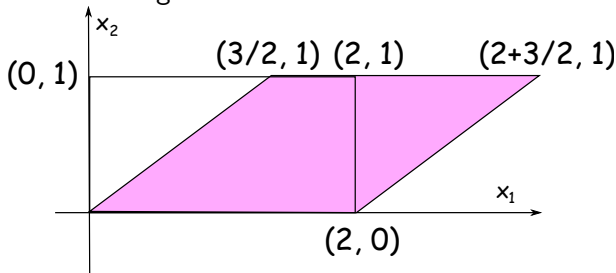
$$T(\vec{x}) = A\vec{x} = \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 + \frac{3}{2} \cdot 0 \\ 0 \cdot x_1 + 1 \cdot 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$$

So T is the identity on horizontal vectors.

For any **vertical** vector $\vec{x} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix}$

$$T(\vec{x}) = A\vec{x} = \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \cdot 0 + \frac{3}{2} \cdot x_2 \\ 0 \cdot 0 + 1 \cdot x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{3}{2}x_2 \\ 0 \end{bmatrix}$$

So a vertical vector is pushed perfectly horizontally, a distance $\frac{3}{2}$ times its length:



Example 2, scaling:

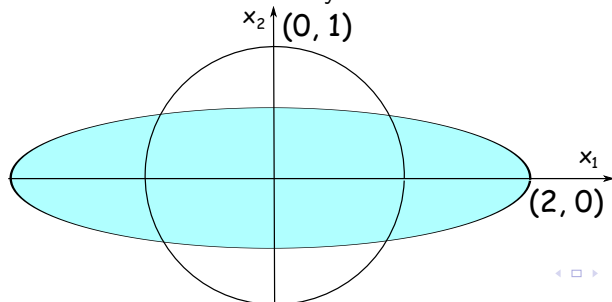
Use

$$A = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

For any vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$T(\vec{x}) = A\vec{x} = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 + 0 \cdot x_2 \\ 0 \cdot x_1 + \frac{1}{2} \cdot x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ \frac{x_2}{2} \end{bmatrix}$$

So T **stretches** horizontally and **contracts** vertically:



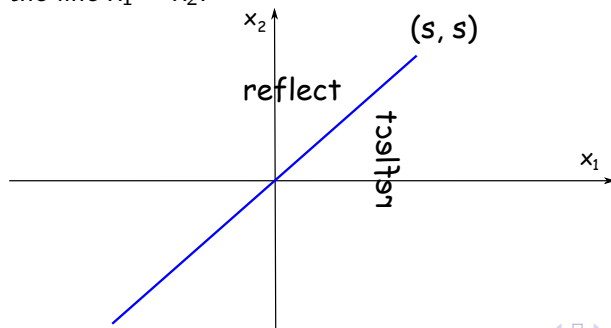
Example 3, reflection through a line:

Use

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0x_1 + 1 \cdot x_2 \\ 1 \cdot x_1 + 0 \cdot x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$$

So T **exchanges** the two coordinates. Looks like reflection through the line $x_1 = x_2$:



Example 4, rotation:

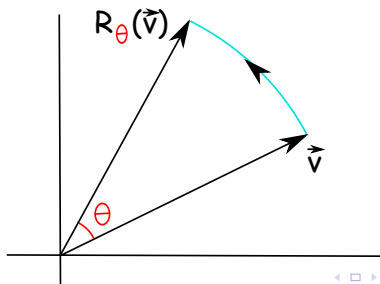
Use

$$A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$$

So horizontal unit vector is rotated c-clockwise an angle θ .

Similarly, for the vertical unit vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, so all of plane rotates:



Some types of problems that can come up:

Question: Suppose $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a **matrix** linear transformation. Suppose A is the matrix of T and $\vec{u} \in \mathbb{R}^n$ is given. What is $T(\vec{u})$?

Answer: Just do matrix-vector multiplication $A\vec{u}$. The result is a vector in $\mathbb{R}^m$.

Question: Suppose $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a **matrix** linear transformation. Suppose A is the matrix of T and $\vec{b} \in \mathbb{R}^m$ is given. Find a vector $\vec{u}$ so that

$$T(\vec{u}) = \vec{b}.$$

Answer: Solve the system of equations given by $A\vec{u} = \vec{b}$.

Reminder: There may be

- no solution (which means $\vec{b}$ is not in the image of T) or
- exactly one solution or
- infinitely many solutions

iClicker question

Suppose T is a matrix linear transformation with matrix A below, and we are seeking all vectors $\vec{u}$ so that $T(\vec{u}) = \vec{b}$.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 0 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 6 \\ 7 \\ 8 \end{bmatrix}$$

How many solutions are there?

- A) Zero.
- B) One
- C) Infinity
- D) Can't tell

What if $\vec{b} = \begin{bmatrix} 6 \\ 7 \\ 0 \end{bmatrix}$?

Question: Describe **all** vectors $\vec{u}$ so that $T(\vec{u}) = \vec{b}$.

Answer: The same approach. Could be no $\vec{u}$, could be exactly one $\vec{u}$, or could be infinitely many such $\vec{u}$'s.

Recall the idea: row reduce the **augmented** matrix to **merely echelon** form.

- Augmentation column is pivot column $\iff$ **no** solutions.
- Augmentation column is **only** non-pivot column $\iff$ **unique** solution.
- There are free variables $\iff$ infinitely many solutions.

If there are solutions, **reduced echelon** form makes it easy to describe them.

Question: Which linear transformations $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ are matrix linear transformations? That is, which come from matrices?

Answer: All of them!

Question: Given a linear transformations $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ how do you find the matrix?

Underlying idea:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \dots x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

So to find

$$T(\vec{x}) = x_1 T \left(\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right) + x_2 T \left(\begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \right) + \dots x_n T \left(\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right)$$

only need to know each $T(\vec{e}_j)$ where

$$\vec{e}_j = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow j^{th} \text{ entry} \quad \text{Let } \vec{a}_j = T(\vec{e}_j)$$

$$T(\vec{x}) = x_1 \vec{a}_1 + x_2 \vec{a}_2 + \dots x_n \vec{a}_n$$

Answer: In matrix notation this is written:

$$T(\vec{x}) = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

That is, the matrix

$$A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \end{bmatrix}$$

is the matrix of T !

Recap: Since T is a linear transformation,

$$T(\vec{x}) = T(x_1\vec{e}_1 + x_2\vec{e}_2 + \dots x_n\vec{e}_n) = x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2) + \dots x_n T(\vec{e}_n) =$$

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \dots x_n\vec{a}_n = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = A\vec{x}.$$

Sample application: Suppose $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a linear transformation so that

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 2 \end{bmatrix}; \quad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

What is $T\left(\begin{bmatrix} -1 \\ 7 \end{bmatrix}\right)$?

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 2 \end{bmatrix}; \quad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

What is $T\left(\begin{bmatrix} -1 \\ 7 \end{bmatrix}\right)$?

One approach (underlying idea):

$$T\left(\begin{bmatrix} -1 \\ 7 \end{bmatrix}\right) = T\left(-1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 7 \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = -1 T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + 7 T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = -1 \begin{bmatrix} 5 \\ 2 \end{bmatrix} + 7 \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Universal approach: (works for any vector)

Matrix of T is $A = \begin{bmatrix} 5 & 3 \\ 2 & 4 \end{bmatrix}$

So

$$T\left(\begin{bmatrix} -1 \\ 7 \end{bmatrix}\right) = \begin{bmatrix} 5 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 7 \end{bmatrix} = -1 \begin{bmatrix} 5 \\ 2 \end{bmatrix} + 7 \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$