

Fun things to do with matrices

Math 4A – Scharlemann

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Question: Which of the following best describes your opinion of the midterm “lockdown” period. That is, the period in which early departure is forbidden, because it is disruptive.

- A) The last ten minutes seemed about right.
- B) Please extend it to 25 minutes (the last half of the exam)
- C) I would prefer no early departures: lockdown of 50 minutes.
- D) Whatever...I don't really care.
- E) We had a midterm?

Let A be an $m \times n$ matrix, that is A has m rows and n columns:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Entry in i^{th} row, j^{th} column of a matrix A denoted $(A)_{ij}$.

So for the matrix above, $(A)_{ij} = a_{ij}$.

The **diagonal entries** are those in which $i = j$.

So for the matrix A above,

$$\{a_{11}, a_{22}, \dots, a_{mm}\}$$

are the diagonal entries (if $m \leq n$; if $n < m$ must stop at a_{nn} .)

A **square matrix** has as many rows as columns (i. e. $m = n$.)

A **diagonal matrix** is square and all **off-diagonal** terms are zero.

So A is diagonal $\iff$ whenever $i \neq j$, the entry $(A)_{ij} = 0$.

$$\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{mm} \end{bmatrix}$$

Also: **upper triangular** and **lower triangular**:

$$\begin{bmatrix} a_{11} & * & \cdots & * \\ 0 & a_{22} & \cdots & * \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{mm} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ * & a_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ * & * & \cdots & a_{mm} \end{bmatrix}$$

Can add two matrices **of the same size!** entry by entry.

More formally: Suppose A and B are **both $m \times n$** matrices. Define $A + B$ by

$$(A + B)_{ij} = (A)_{ij} + (B)_{ij},$$

In other words, the

ij entry of $A + B$ is the sum of the ij entry of A with the ij entry of B .

So, for example

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & 4 & -2 \end{bmatrix} + \begin{bmatrix} 7 & 5 & -3 \\ 3 & 1 & -4 \end{bmatrix} = \begin{bmatrix} 8 & 7 & -4 \\ 6 & 5 & -6 \end{bmatrix}$$

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Question: What's the sum of these two matrices:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix} = ??$$

A) $\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$

B) $\begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}$

C) $\begin{bmatrix} 5 & 4 \\ 6 & 5 \end{bmatrix}$

D) $\begin{bmatrix} 5 & 6 \\ 4 & 5 \end{bmatrix}$

E) I'm confused.

Similarly, can multiply any matrix by a real number, sometimes called a **scalar**. This is called **scalar multiplication** of the matrix. Again done entry by entry.

More formally: Suppose A is a matrix and $t \in \mathbb{R}$. Define tA by

$$(tA)_{ij} = t(A)_{ij}.$$

In other words, the ij entry of tA is just t times the ij entry of A .

So, for example,

$$7 \begin{bmatrix} 1 & 2 & -1 \\ 3 & 4 & -2 \end{bmatrix} = \begin{bmatrix} 7 & 14 & -7 \\ 21 & 28 & -14 \end{bmatrix}$$

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Question: Calculate

$$-1 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + 3 \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix} = ??$$

A) $\begin{bmatrix} 11 & 6 \\ 4 & -1 \end{bmatrix}$

B) $\begin{bmatrix} 11 & 4 \\ 6 & -1 \end{bmatrix}$

C) $\begin{bmatrix} 11 & 4 \\ 1 & 6 \end{bmatrix}$

D) $\begin{bmatrix} 11 & 4 \\ -1 & 6 \end{bmatrix}$

E) I'm confused.

Why emphasize **scalar** multiplication? Because there is another kind of multiplication called **matrix** multiplication:

We have already defined matrix-vector multiplication $A\vec{x}$. If we

think of the vector $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ as an $n \times 1$ matrix then $A\vec{x}$ was

already an example of matrix multiplication.

More generally, suppose A is an $m \times n$ matrix and B is the $n \times p$ matrix

$$B = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_p \end{bmatrix}$$

then AB is the $m \times p$ matrix given by:

$$AB = \begin{bmatrix} A\vec{b}_1 & A\vec{b}_2 & \cdots & A\vec{b}_p \end{bmatrix}$$

Translation:

To calculate $(AB)_{ij}$, multiply the i^{th} row of A with the j^{th} column of B :

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & \cdots & b_{2j} & \cdots & b_{2p} \\ \vdots & \vdots & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{np} \end{bmatrix} =$$

$$\begin{bmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1p} \\ \vdots & \vdots & \vdots & & \vdots \\ c_{i1} & \cdots & c_{ij} & \cdots & c_{ip} \\ \vdots & \vdots & \vdots & & \vdots \\ c_{m1} & \cdots & c_{mj} & \cdots & c_{mp} \end{bmatrix}; \quad c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots a_{in}b_{nj}$$

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Question: Calculate

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix} = ??$$

A) $\begin{bmatrix} 10 & 24 \\ 4 & 10 \end{bmatrix}$

B) $\begin{bmatrix} 4 & 10 \\ 10 & 24 \end{bmatrix}$

C) $\begin{bmatrix} 10 & 4 \\ 10 & 24 \end{bmatrix}$

D) $\begin{bmatrix} 10 & 4 \\ 24 & 10 \end{bmatrix}$

E) I'm confused.

Major warning: Matrix multiplication is **not commutative**. That is, in general, $AB \neq BA$

Example 1: Let

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}; \quad B = \begin{bmatrix} 2 & 0 & -4 \\ 5 & -2 & 6 \end{bmatrix}$$

Then

$$AB = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 & -4 \\ 5 & -2 & 6 \end{bmatrix} = \begin{bmatrix} 17 & -6 & 14 \\ -1 & 2 & -14 \end{bmatrix}$$

whereas

$$BA = \begin{bmatrix} 2 & 0 & -4 \\ 5 & -2 & 6 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} = \text{can't do it}$$

But having the wrong shape for multiplication is not the only problem: $AB \neq BA$ even when A, B are 2×2 matrices:

Example 2: Let

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}; \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Then

$$AB = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix}$$

whereas

$$BA = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & -1 \end{bmatrix} \neq \begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix} = AB$$

Other strange properties:

- **Can't** cancel: If $AB = AC$ it does not follow that $B = C$, even when $A \neq 0$
- **Can** multiply to zero: If $AB = 0$ it does **not follow** that $A = 0$ or $B = 0$

Example of both: Let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} 0 & 12 \\ 0 & 0 \end{bmatrix}; \quad C = \begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix}.$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 12 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix}$$

But mostly the properties are nice:

Theorem

If A , B , and C are sized so that the products are defined, and $t \in \mathbb{R}$, then

- ① $A(BC) = (AB)C$ (associative),
- ② $A(B + C) = AB + AC$ (distributes),
- ③ $(B + C)A = BA + CA$ (distributes),
- ④ $t(AB) = (tA)B = A(tB)$ (associative)

Only the first rule is hard to see.

Application to linear transformations: Let's apply the first rule $A(BC) = (AB)C$ to the case $C =$ a vector $\vec{x}$:

$$A(B\vec{x}) = (AB)\vec{x}$$

Consider the linear transformations T_A and T_B given by A and B :

$$T_B(\vec{x}) = B\vec{x} \quad \text{and} \quad T_A(B\vec{x}) = A(B\vec{x}).$$

Combining these, we get

$$A(B\vec{x}) = T_A(T_B(\vec{x})) = (T_A \circ T_B)(\vec{x}).$$

Here $T_A \circ T_B$ means the composition: do T_B first, then T_A second. So we get, for any vector $\vec{x}$,

$$(T_A \circ T_B)(\vec{x}) = A(B\vec{x}) = (AB)\vec{x}$$

So the amazing upshot:

Corollary

The product matrix AB defines the linear transformation $T_A \circ T_B$.

Definition

The $n \times n$ diagonal matrix

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

is called the **identity matrix**.

Easy to check:

- For A any $m \times n$ matrix, $AI_n = A$.
- For B any $n \times p$ matrix, $I_n B = B$.
- So if A is a square $n \times n$ matrix then $I_n A = A = AI_n$.

Definition

Let A be an $m \times n$ matrix. The **transpose** of A is the $n \times m$ matrix, denoted A^T , such that

$$(A^T)_{ij} = (A)_{ji}.$$

In other words, taking the transpose **switches** rows and columns:
 i^{th} row becomes i^{th} column and j^{th} column becomes j^{th} row:

Example:

$$\begin{bmatrix} 2 & 0 & -4 \\ 5 & -2 & 6 \end{bmatrix}^T = \begin{bmatrix} 2 & 5 \\ 0 & -2 \\ -4 & 6 \end{bmatrix}$$

Theorem

For any matrices A and B (of the same size) and $s \in \mathbb{R}$, we have

- ① $(A^T)^T = A$,
- ② $(A + B)^T = A^T + B^T$,
- ③ $(sA)^T = sA^T$.
- ④ $(AB)^T = B^T A^T$. (Order of multiplication reversed!)

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Question: We saw earlier that

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}; \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \implies AB = \begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix} \quad BA = \begin{bmatrix} 3 & 2 \\ 2 & -1 \end{bmatrix}$$

Using the relations above, what is

$$A^T B^T = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

A) $\begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix}$

B) $\begin{bmatrix} 3 & 2 \\ 2 & -1 \end{bmatrix}$

C) $\begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix}$

D) $\begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}$