

# AUTOMORPHISMS OF THE 3-SPHERE PRESERVING A GENUS TWO HEEGAARD SPLITTING

MARTIN SCHARLEMANN

ABSTRACT. We give an updated proof of a 1933 theorem of Goeritz, exhibiting a finite set of generators for the the group of automorphisms of  $S^3$  that preserve a genus two Heegaard splitting.

## 1. INTRODUCTION

In 1933 Goeritz [Go] described a set of automorphisms of the standard unknotted genus two handlebody in  $S^3$ , each of which extends to all of  $S^3$ . He further observed that any such automorphism is a product of elements of this finite set. Stated somewhat differently, Goeritz identified a finite set of generators for the group  $\mathcal{H}$ , defined as isotopy classes of orientation-preserving homeomorphisms of the 3-sphere that leave a genus two Heegaard splitting invariant. Goeritz' theorem was generalized to Heegaard splittings of arbitrarily high genus by Powell [Po], but the proof contains a serious gap.<sup>1</sup> So a foundational question remains unresolved: Is the group of automorphisms of the standard genus  $g$  Heegaard splitting of  $S^3$  finitely generated and, if so, what is a natural set of generators. The finite set of elements that Powell proposes as generators remains a very plausible set.

Since the gap in Powell's proof has escaped attention for 25 years, Goeritz' original theorem might itself be worth a second look. His argument is difficult for the modern reader to follow, is published in a fairly inaccessible journal and is a bit old-fashioned in its outlook. In view of the use that has been made of it in recent work on tunnel number one knots (cf [ST], [Sc]) it seems worthwhile to present an updated proof, in hopes also that it might be relevant to the open analogous problem for Heegaard splittings of higher genus.

The purpose of this note is to present such a proof, one influenced by the idea of thin position. One way to describe the outcome of this

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<sup>1</sup>On p. 210, Case 2 the argument requires that, among the chambers into which  $\phi^{-1}(s_k)$  divides the handle, there are two adjacent ones that each contain pieces of  $G_k^h$ . There is no reason presented why this should be true.

investigation is this: there is a natural 2-complex  $\Gamma$  (which deformation retracts to a graph) on which  $\mathcal{H}$  acts transitively. One can write down an explicit finite presentation for the stabilizer  $\mathcal{H}_P$  of a vertex  $v_P \in \Gamma$  and observe that the stabilizer acts transitively on the edges of  $\Gamma$  incident to  $v_P$ . In particular, if we add to  $\mathcal{H}_P$  any element  $\delta$  of  $\mathcal{H}$  that takes  $v_P$  to some adjacent vertex then the subgroup generated by  $\mathcal{H}_P$  and  $\delta$  is exactly the subgroup that preserves the component in which  $v_P$  lies. This in fact is all of  $\mathcal{H}$ , because it turns out that  $\Gamma$  is connected. The proof that  $\Gamma$  is connected can be viewed as the core argument in this paper.

## 2. THE COMPLEX $\Gamma$ AND ITS VERTEX STABILIZERS

We outline the general setting, referring the reader to [Po, Section 1] for details. Let  $V$  denote the standard unknotted genus-two handlebody in  $S^3$ , with closed complement  $W$  also a genus two handlebody. Let  $\mathcal{H}$  denote the group of orientation-preserving homeomorphisms of  $S^3$  that preserve  $V$ . Regard two as equivalent if there is an isotopy from one to the other via isotopies that preserve  $V$ . Any orientation preserving homeomorphism of  $S^3$  is isotopic to the identity, so an element  $h : (S^3, V) \rightarrow (S^3, V)$  of  $\mathcal{H}$  is isotopic, as a homeomorphism of  $S^3$ , to the identity. This gives an alternate view of  $\mathcal{H}$ : an element of  $\mathcal{H}$  corresponds to an isotopy of  $S^3$  from the identity to a homeomorphism that preserves  $V$  setwise.

For  $T = \partial V = \partial W$ ,  $S^3 = V \cup_T W$  is a genus two Heegaard splitting of  $S^3$ . In the language of Heegaard splittings, a *reducing sphere*  $P \subset S^3$  is a sphere that intersects  $T$  transversally in a single essential circle and so intersects each handlebody in a single essential disk. Since  $P$  is separating in  $S^3$ ,  $P \cap T$  is a separating curve in  $T$ , which we will denote  $c$ . A straightforward innermost disk argument shows that  $P$  is determined up to isotopy rel  $T$  by the circle  $c$ .

Suppose  $Q$  is another reducing sphere, with the circles  $c$  and  $Q \cap T$  isotoped to intersect transversally and minimally in  $T$ . Then the number of points of intersection  $|P \cap T \cap Q|$  is denoted  $P \cdot Q$ . Clearly  $P \cdot Q = 0$  if and only if  $P$  and  $Q$  are isotopic since the only separating essential curve in either punctured torus component of  $T - c$  is boundary parallel. Since reducing spheres are separating,  $P \cdot Q$  is always even. An elementary argument (see [ST, Lemma 2.5]) shows that  $P \cdot Q \neq 2$  and in some sense characterizes (up to multiple half-Dehn twists about  $c$ ) all spheres  $Q$  so that  $P \cdot Q = 4$ . See Figure 1

This suggests a useful simplicial complex: Let  $\Gamma$  be the complex in which each vertex represents an isotopy class of reducing spheres and

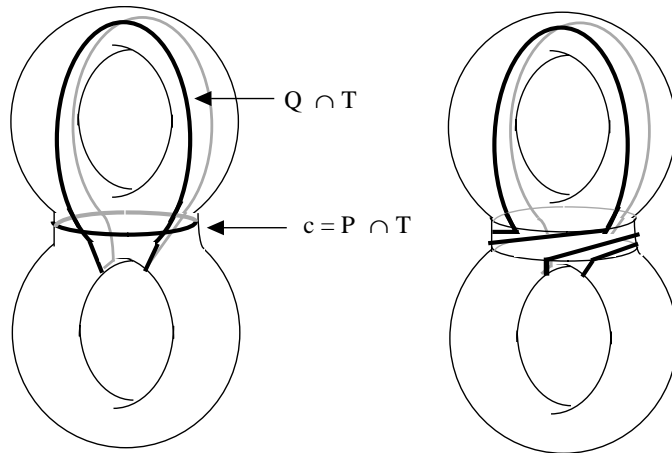


FIGURE 1

a collection  $P_0, \dots, P_n$  of reducing spheres bounds an  $n$ -simplex if and only if  $P_i \cdot P_j = 4$  for all  $0 \leq i \neq j \leq n$ . In fact it follows easily from the characterization in [ST, Lemma 2.5] that  $n \leq 2$ . Figure 2 illustrates a collection of three reducing spheres whose corresponding vertices in  $\Gamma$  span a 2-complex in  $\Gamma$ . (An alternate view, in which  $V$  appears as *(pair of pants)*  $\times I$ , is shown in Figure 3.) Thus  $\Gamma$  is a 2-complex.

Each edge of  $\Gamma$  lies on a single 2-simplex. This is perhaps best seen in Figure 3: The curve  $P \cap T$  is uniquely defined by the curves  $Q \cap T$  and  $R \cap T$  shown. (For example, if the curve  $P \cap T$  is altered by Dehn twists around the outside boundary of the pair of pants, it becomes a curve that is non-trivial in  $\pi_1(V)$ , so it can't bound a disk in  $V$ .) So the 2-complex  $\Gamma$  deformation retracts naturally to a graph, in which each 2-simplex in  $\Gamma$  is replaced by the cone on its three vertices.

A reducing sphere  $P$  divides  $S^3$  into two 3-balls  $B_{\pm}$  and  $T$  intersects each 3-ball in a standard unknotted punctured torus, unique up to isotopy rel boundary. It follows that for any two reducing spheres  $P$  and  $Q$  there is an orientation preserving homeomorphism of  $S^3$ , preserving  $V$  as a set, that carries  $P$  to  $Q$ . Thus  $\mathcal{H}$  acts transitively on the vertices of  $\Gamma$ .

We now explicitly give a presentation of the group that stabilizes a vertex of  $\Gamma$ . As above, let  $P$  be a reducing sphere for the Heegaard splitting  $S^3 = V \cup_T W$  and suppose  $h : (S^3, V) \rightarrow (S^3, V)$  is an orientation preserving homeomorphism that leaves  $P$  invariant. That is, suppose  $h$  represents an element in  $\mathcal{H}$  that stabilizes the vertex in  $\Gamma$  corresponding to  $P$ .

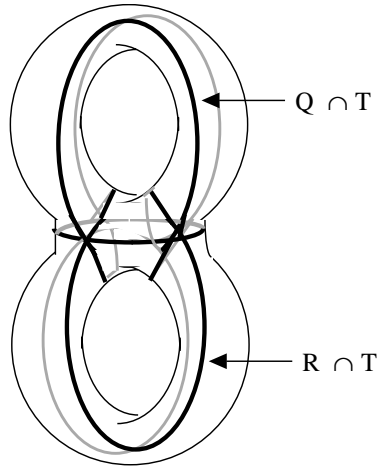


FIGURE 2

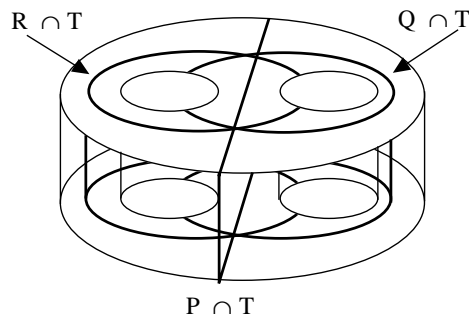


FIGURE 3

First assume that  $h$  preserves the orientation of  $P$ . Let  $T_{\pm} = T \cap B_{\pm}$  denote the two punctured torus components of  $T - P$ ; since  $h$  preserves orientation of both  $S^3$  and  $P$  we have  $h(T_+) = T_+$  and  $h(T_-) = T_-$ . Up to isotopy there is a unique non-separating curve  $\mu_{\pm} \subset T_{\pm}$  that bounds a disk in  $V$  and a unique non-separating curve  $\lambda_{\pm}$  that bounds a disk in  $W$  and we may choose these curves so that  $\mu_{\pm} \cap \lambda_{\pm}$  is a single point. Hence, up to equivalence in  $\mathcal{H}$ , we may with little difficulty assume that each wedge of circles  $\gamma_{\pm} = \mu_{\pm} \cup \lambda_{\pm}$  is mapped to itself by  $h$  and, on each  $\gamma_{\pm}$ , the homeomorphisms  $h|_{\mu_{\pm}} : \mu_{\pm} \rightarrow \mu_{\pm}$  and  $h|_{\lambda_{\pm}} : \lambda_{\pm} \rightarrow \lambda_{\pm}$  are either simultaneously orientation preserving (in which case we can take them both to be the identity) or simultaneously orientation reversing (in which case we can take them each to be reflections that preserve their intersection point). Having identified  $h$  on  $\gamma_{\pm}$  we observe that  $T - (\gamma_+ \cup \gamma_-)$  is an annulus  $A$ , and any end-preserving homeomorphism

$A \rightarrow A$  is determined up to isotopy and Dehn twists around its core by  $h|_{\partial A}$ . The upshot of this discussion is the following description:

**Lemma 1.** : *Let  $\mathcal{H}_P^+$  be the subgroup of  $\mathcal{H}$  represented by homeomorphisms that restrict to orientation-preserving homeomorphisms of  $P$ . Then*

$$\mathcal{H}_P^+ \cong Z_2 + Z$$

with generators given by the automorphisms  $\alpha$  and  $\beta$  shown in Figure 4.

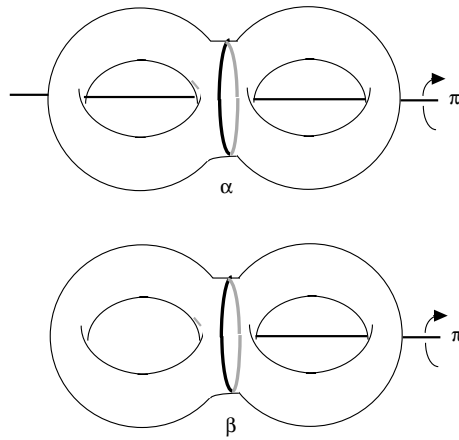


FIGURE 4

The situation is only slightly more complicated if we drop the requirement that  $h|_P$  be orientation preserving since the order two element  $\gamma \in \mathcal{H}$  shown in Figure 5 preserves  $P$  but reverses its orientation.

**Lemma 2.** *Let  $\mathcal{H}_P$  be the subgroup of  $\mathcal{H}$  represented by homeomorphisms that preserve  $P$ . Then  $\mathcal{H}_P$  is an extension of  $\mathcal{H}_P^+$  by  $Z_2$ , via the relations  $\gamma\alpha\gamma = \alpha$  and  $\gamma\beta\gamma = \alpha\beta$ .*

Finally, observe that if  $Q$  and  $Q'$  are reducing sphere so that  $P \cdot Q = 4$  and  $P \cdot Q' = 4$  then for some  $n \in \mathbb{Z}$ , either  $\beta^n$  or  $\beta^n \gamma$  carries  $Q$  to  $Q'$ . (See discussion of Figure 1 above.) Interpreting this in terms of the action of  $\mathcal{H}$  on the complex  $\Gamma$  we have:

**Corollary 3.** *Let  $\mathcal{H}_P$  be the subgroup of  $\mathcal{H}$  that stabilizes the vertex  $v_P \in \Gamma$  corresponding to  $P$ . Then  $\mathcal{H}_P$  is transitive on the edges of  $\Gamma$  incident to  $v_P$ .*

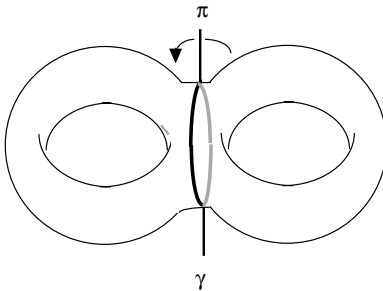


FIGURE 5

## 3. INTERSECTION OF REDUCING SPHERES

Suppose  $T_0$  is an oriented punctured torus containing oriented simple closed curves  $\mu, \lambda$  that intersect in a single point. For  $\alpha$  an essential embedded arc in  $T_0$  define the *slope*  $\sigma(\alpha) \in \mathbb{Q} \cup \{\infty\}$  of the arc  $\alpha$  as follows: Orient  $\alpha$  and let  $p = \alpha \cdot \mu$  and  $q = \alpha \cdot \lambda$  be the algebraic intersection numbers of the corresponding homology classes. Then  $\sigma(\alpha) = p/q$ . Reversing the orientation of  $\alpha$  has no effect on the slope, since it changes the sign of both  $p$  and  $q$ . An alternate description of the (unsigned) slope is this: minimize by an isotopy in  $T_0$  the numbers  $p = \alpha \cap \mu$  and  $q = \alpha \cap \lambda$ ; then  $|\sigma(\alpha)| = p/q$ . If  $\beta \subset T_0$  is another essential arc, with slope  $r/s$  define their distance  $\Delta(\alpha, \beta) = |ps - qr| \in \mathbb{N}$ . It is easy to see that if the arcs  $\alpha$  and  $\beta$  are disjoint then  $\Delta(\alpha, \beta) \leq 1$ . Any embedded collection of arcs in  $T_+$  constitutes at most three parallel families of arcs, with slopes of any pair of disjoint non-parallel arcs at a distance of one.

We now apply this terminology in the setting given above:  $P$  is a reducing sphere for  $V \cup_T W$ , the closed 3-ball components of  $S^3 - P$  are  $B_{\pm}$ , the punctured tori  $T \cap B_{\pm}$  are denoted  $T_{\pm}$  and  $Q$  is a reducing sphere for  $V \cup_T W$  that is not isotopic to  $P$  and has been isotoped so as to minimize  $|P \cap Q \cap T| = P \cdot Q$ . It will be convenient to imagine  $P$  as a level sphere of a standard height function on  $S^3$ , with  $B_+$  above  $P$  and  $B_-$  below  $P$ . When we use the terms *above* and *below* in what follows, we will be referring to such a height function.

In each of  $T_{\pm}$  there are closed non-separating curves  $\mu_{\pm}, \lambda_{\pm}$  bounding respectively disks in  $V$  and disks in  $W$  and for each pair,  $\mu_{\pm} \cap \lambda_{\pm}$  is a single point. We will consider the collection of arcs  $Q \cap T_{\pm}$  and their slopes with respect to  $\mu_{\pm}, \lambda_{\pm}$ . Fix at the outset some orientations, e.g. orient  $T$  (hence  $T_{\pm}$ ) as  $\partial V$  and orient  $\mu_{\pm}, \lambda_{\pm}$  so that the algebraic intersection number  $\mu_{\pm} \cdot \lambda_{\pm} = 1$ . (The exact choice of orientations is not critical.)

**Lemma 4.** *There is some arc in either  $Q \cap T_+$  or in  $Q \cap T_-$  that is of slope  $\infty$  and another such arc is of slope 0.*

*Proof.* An outermost disk cut off by the disk  $P \cap V$  from the disk  $Q \cap V$  is a meridian disk  $D$  of the solid torus  $V \cap B_+$  or  $V \cap B_-$ . Then the arc  $D \cap T$  must be of slope 0. A symmetric argument on the disks  $P \cap W, Q \cap W$  gives an arc of slope  $\infty$ .  $\square$

**Lemma 5.** *Suppose that an arc  $\alpha_+$  of  $Q \cap T_+$  has slope  $\infty$  (resp. 0) and that there is an arc  $\alpha_-$  of slope 0 (resp.  $\infty$ ) in  $T_-$  that is disjoint from  $Q$ . Then there is a reducing sphere  $R$  so that  $P \cdot R = 4$  and  $R \cdot Q < P \cdot Q$ .*

*The same hypothesis, but with  $T_+$  and  $T_-$  reversed, leads to the same conclusion.*

*Proof.* Since  $\alpha_-$  is merely required to be disjoint from  $Q$ , with no loss we may assume that the ends of  $\alpha_{\pm}$  on the circle  $c = P \cap T$  are disjoint. Say that the arcs  $\alpha_{\pm}$  cross if the ends of  $\alpha_+$  and  $\alpha_-$  alternate around  $c$ ; that is, if the ends of  $\alpha_+$  lie on different arc components of  $c - \alpha_-$ .

**Claim:** Some pair of arcs that satisfy the hypotheses for  $\alpha_{\pm}$  cross.

**Proof of claim:** Assume, on the contrary, that no such pair of arcs crosses. Then among arcs of  $Q \cap T_{\pm}$  satisfying the conditions for  $\alpha_{\pm}$  choose the pair whose ends are closest to each other on the circle  $c$ . The ends of  $\alpha_{\pm}$  divide  $c$  into four arcs, one of them, denoted  $\beta_+$ , is bounded by the ends of  $\alpha_+$  and the other, denoted  $\beta_-$ , by the ends of  $\alpha_-$ . Let  $c_{\pm} = |Q \cap \beta_{\pm}|$ .

$T_+ - \eta(\alpha_+)$  is an annulus  $A$ ; denote the boundary component that contains  $\beta_{\pm}$  by  $\partial_{\pm}A$ . Then  $|\partial_-A \cap Q| = c_+$  and  $|\partial_-A \cap Q| \geq c_-$ . (The inequality reflects the fact that  $Q$  may also intersect the two intervals  $c - \beta_{\pm}$ .) No arc of  $Q \cap A$  can have both ends on  $\partial_-A$ , else it would have been parallel to  $\alpha_+$  in  $T_+$ , and yet closer to  $\alpha_-$ . We conclude that  $c_+ \geq c_-$ . Arguing symmetrically on  $T_- - \eta(\alpha_-)$ , we obtain  $c_- \geq c_+ + 2$ , the extra 2 arising from the ends of  $\alpha_+ \subset Q - c_+$ . The two inequalities conflict, a contradiction proving the claim.

Following the claim, we assume that  $\alpha_{\pm}$  cross. Let  $\rho \subset T$  be the circle obtained by banding the circle  $c$  to itself along the two arcs  $\alpha_{\pm} \subset T_{\pm}$ . It is a single circle because  $\alpha_{\pm}$  cross. Moreover, it's easy to see that  $\rho$  is an essential circle in  $T$  (there are essential curves in  $T$  on both sides of  $\rho$ ) and that  $\rho$  bounds disks both in  $V$  and  $W$ . So  $\rho$  is the intersection with  $T$  of a reducing sphere  $R$ . Moreover,  $R \cdot P = |\rho \cap P| = 4$  and  $R \cdot Q \leq |\rho \cap Q| = |c \cap Q| - 2 \leq P \cdot Q - 2$  since the ends of  $\alpha_+$  no longer count.  $\square$

**Proposition 6.** *There is a reducing sphere  $R$  so that  $P \cdot R = 4$  and  $R \cdot Q < P \cdot Q$ .*

*Proof.* If there are two arcs of  $(Q \cap T) - c$ , one of slope 0 and one of slope  $\infty$ , one lying in  $T_+$  and the other lying in  $T_-$ , the result follows from Lemma 5. Following Lemma 4 we know that there arcs of slope both 0 and  $\infty$ . Thus we are done unless both these arcs lie in  $T_-$  say, and each arc of  $Q \cap T_+$  has finite, non-zero slope. Moreover, if all arcs of  $Q \cap T_+$  have slope 1 (or slope  $-1$ ) then a curve of slope 0 in  $T_+$  will be disjoint from  $Q \cap T_+$  and again we would be done by Lemma 5. If  $\sigma, \tau$  are slopes of arcs in  $Q \cap T_+$ , then, because  $|\Delta(\sigma, \tau)| \leq 1$ , the inequality  $0 < |\sigma| < 1$  would imply that  $|\tau| \leq 1$  and that  $\sigma$  and  $\tau$  have the same sign. Finally, a curve that has slope  $\sigma$ , will have slope  $1/\sigma$  if the roles of  $V$  and  $W$  are reversed. Following these considerations, we may as well restrict to the following case:

- Both slopes 0 and  $\infty$  arise among the arcs of  $Q \cap T_-$  and
- all arcs of  $Q \cap T_+$  have slope  $\sigma$  with  $0 < \sigma \leq 1$  and not all have slope 1.

Now consider a sphere  $P^+ \subset B_+$  that intersects the solid torus  $V \cap B_+$  in two meridian disks, and so intersects  $W$  in an annulus. Again isotope the curve  $Q \cap T$  so that it intersects the two meridian circles  $P^+ \cap T$  minimally. Any arc of  $Q \cap T_+$  must intersect  $P^+$ , else the arc would be of slope 0. In particular, there is an essential non-separating disk  $F \subset W$  so that  $\partial F \subset T_+$  (i. e.  $\partial F$  is a longitude of the solid torus  $V \cap B_+$ ) so that  $F \cap P^+$  is a single spanning arc of the annulus  $P^+ \cap W$  and so that the arc of  $\partial F - P^+$  lying below  $P^+$  (i. e. in the pair of pants component of  $T_+ - P^+$  adjacent to  $c$ ) is disjoint from  $Q$ . See Figure 6

We now examine outermost disks cut off from the disk  $Q \cap W$  by the annulus  $P^+ \cap W$ . Let  $E$  be any such disk. Let  $V^\pm$  be the closed components of  $V - P^+$ , with  $V^+$  the 1-handle lying above  $P^+$  and  $V_-$  the solid torus lying below  $P^+$ .

**Claim 1:** The outermost arc  $\epsilon = \partial E \cap P^+$  spans the annulus  $P^+ \cap W$ .

**Proof of Claim 1:** This is obvious if  $E$  lies above  $P^+$ , since all arcs of  $Q \cap T$  above  $P^+$  span the 1-handle  $V^+$ . If  $E$  lies below  $P^+$  the argument is a bit more subtle. Note that  $V^-$  is a solid torus with two disks  $d_1, d_2$  in  $\partial V^-$  attached to  $P^+$ . A simple counting argument (the  $d_i$  are parallel in  $V^+$ ) shows that any arc of  $Q \cap (\partial V^- - P^+)$  that has both ends on the same disk  $d_i$  is essential in the torus  $\partial V^-$ . So an outermost disk  $D \subset V^-$  cut off from the disk  $Q \cap V$  by the meridian disks  $d_i$  must be a meridian disk of the solid torus  $V^-$ , with both ends



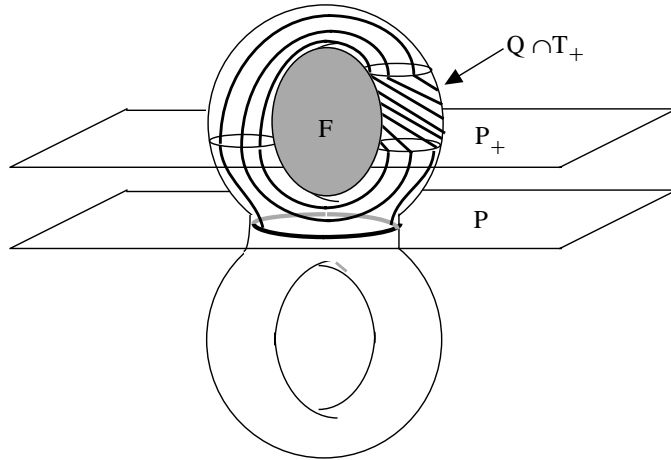


FIGURE 6

on  $d_1$ , say. The same counting argument shows that some essential arc in  $Q \cap V^-$  must have both its ends on  $d_2$  and so is a meridinal arc for  $V^-$  there as well. If the ends of  $\epsilon$  were both on the same  $d_i$ , then  $\epsilon$  would be a longitudinal arc disjoint from the meridinal arc with ends at the other disk  $d_j, j \neq i$ . But a longitudinal arc and a meridinal arc based at different points must necessarily intersect. Hence the ends of  $\epsilon$  each lie on a different disk  $d_i$ , proving Claim 1.

**Claim 2:** All the outermost disks cut off from  $Q \cap W$  by  $P^+$  must lie on the same side of  $P^+$ .

**Proof of claim 2:** Suppose, on the contrary, that the outermost disks  $E^\pm$  are cut off, with  $E^-$  lying in the component of  $S^3 - P^+$  that lies below  $P^+$  and  $E_+$  lying in the component that lies above  $P^+$ . Following Claim 1, both arcs  $\epsilon^\pm = E^\pm \text{cap} P^+$  span the annulus  $P^+ \cap W$ .

Since the arc  $E_- \cap T$  is disjoint from  $\partial F$  it follows from a simple innermost disk, outermost arc argument, that all of  $E_-$  can be made disjoint from  $F$ ; in particular the spanning arcs  $\epsilon^-$  and  $F \cap P^+$  are disjoint. Since the spanning arc  $\epsilon^+$  is disjoint from the spanning arc  $\epsilon^-$  which in turn is disjoint from the spanning arc  $F \cap P^+$ ,  $\epsilon^+$  can be isotoped off of  $F \cap P^+$  without moving  $\epsilon^-$ . (See Figure 7.) Then again an innermost disk, outermost arc argument allows us to isotope all of  $E_-$  off of  $F$ . Now consider any arc component  $\gamma$  of  $(Q \cap T_+) - P^+$ . If  $\gamma$  lies below  $P^+$  then it is disjoint from  $\partial F$ , by construction; if  $\gamma$  lies above  $P^+$  then since it is disjoint from  $E^+$ , it intersects  $\partial F$  at most once. In particular, any arc of  $Q \cap T^+$  intersects a component of  $P^+ \cap T_+$  at least as often as it intersects  $\partial F$ , hence its slope has absolute

value  $\geq 1$ . This contradicts the second property itemized above, and so proves claim 2).

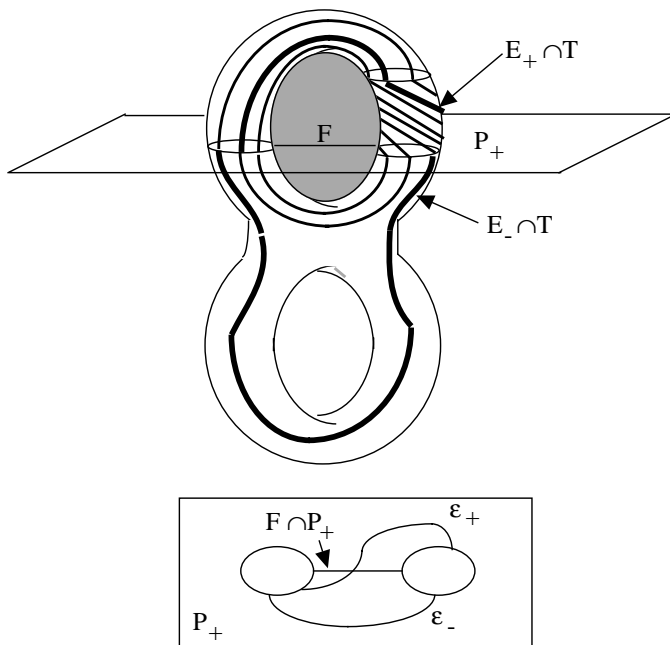


FIGURE 7

**Claim 3:** All the outermost disks cut off from  $Q \cap W$  by  $P^+$  must lie above  $P^+$ .

**Proof of claim 3:** Following claim 2) the alternative would be that they all lie below (in  $B_-$ ). We show how this leads to a contradiction. Consider the disk  $Q \cap W$  and how it is cut up by the annulus  $P^+ \cap W$ . A standard innermost disk argument ensures that all closed curves of intersection can be removed. There is at least one (disk) component  $E_0$  of  $(Q \cap W) - P^+$  that is “second outermost”, i. e. it is adjacent to some  $n \geq 2$  other components of  $(Q \cap W) - P^+$ , all but at most one of them outermost. See Figure 8. Since  $E_0$  is adjacent to an outermost component, all of which we are assuming lie below  $P^+$ ,  $E_0$  must lie above  $P^+$ . By Claim 1), all the outermost arcs of intersection of  $P^+$  with the disk  $Q \cap W$  must span the annulus  $W \cap P^+$ , so it follows that each of the  $n$  arc components of  $\partial E_0 \cap T$  spans the 1-handle  $V^+$ . In particular, the union of the disk  $E_0$  with the punctured solid torus  $P^+ \cup V^+$  is the spine of a Lens space in  $S^3$ , a contradiction proving Claim 3).

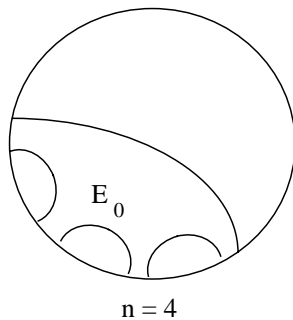


FIGURE 8

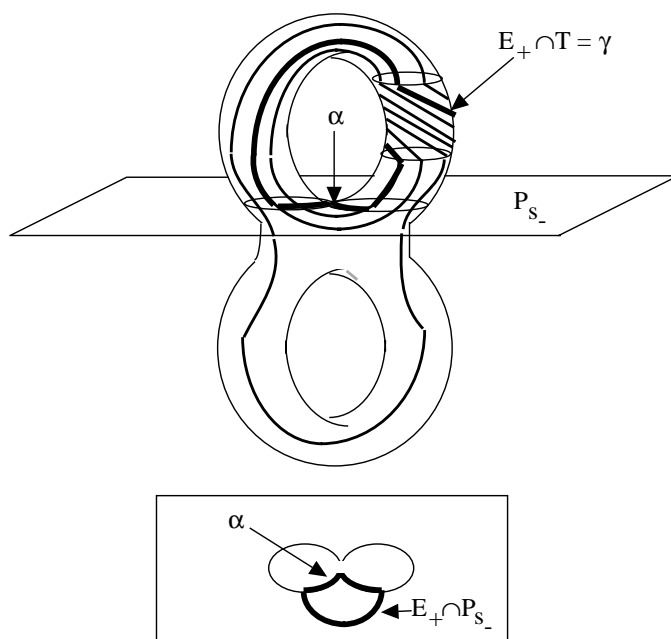


FIGURE 9

Following Claim 3), consider a sphere  $P^s$  that passes through the saddle point of  $T_+$  that lies below  $B^+$ . We can assume (see Figure 9) that  $P^s$  intersects  $Q$  transversally and that every arc of  $Q \cap T^+$  that lies above  $P^s$  spans the 1-handle  $V^+$ . According to claim 3) applied to a plane just slightly higher than  $P^s$ ,  $P^s$  (and so also a plane  $P^{s-}$  lying just below  $P^s$ ) cuts off a disk  $E^+$  from  $Q \cap W$  that lies above the plane. Let  $\alpha \subset (P^{s-} \cap T)$  be an arc parallel in  $P^{s-} \cap W$  to the arc  $E^+ \cap P^{s-}$ , so the union  $\lambda$  of  $\alpha$  and the arc  $\gamma = E^+ \cap T$  is a longitude lying above  $P^{s-}$  (indeed  $\lambda$  is a meridian of  $W$ ). It's easy to isotope the ends of  $\gamma$  closer together in  $\alpha$  until no arc of  $(Q \cap T) - P^{s-}$  lying above  $P^{s-}$  has

more than one end on  $\lambda$ . It then follows just as in the proof of Claim 2) that any arc component of  $Q \cap T_+$  intersects a meridian of  $V^+$  at least as often as it intersects  $\lambda$  and so has slope  $\geq 1$ , a contradiction that completes the proof.  $\square$

**Corollary 7.** *The 2-complex  $\Gamma$  is connected.*

*Proof.* Let  $w$  be a fixed vertex of  $\Gamma$ , with associated reducing sphere  $Q$ . Let  $\Gamma_0$  be any component of  $\Gamma$ . Choose a reducing sphere  $P$  among those represented by vertices in  $\Gamma_0$  so that  $P \cdot Q$  is minimized. Unless  $P = Q$ , Proposition 6 provides a reducing sphere  $R$  which is represented by a vertex in  $\Gamma_0$  (indeed one adjacent to the vertex representing  $P$ ) but for which  $R \cdot Q < P \cdot Q$ . From the contradiction we conclude then that indeed  $P = Q$ , so  $w \in \Gamma_0$ .  $\square$

Corollary 7 is essentially [ST, Proposition 2.6]. There we used Goeritz' theorem to prove the proposition; here we have proven the proposition from first principles and now observe that it proves Goeritz' theorem.

#### 4. A FINITE SET OF GENERATORS

**Theorem 8.** *Suppose  $\delta \in \mathcal{H}$  is any element with the property that  $P \cdot \delta(P) = 4$ . Then the group  $\mathcal{H}$  is generated by  $\alpha, \beta, \gamma, \delta$ .*

*Proof.* Choose any  $h \in \mathcal{H}$  and let  $Q = h(P)$ . If  $Q = P$  the result follows immediately from Lemma 2. Otherwise, following Corollary 7, there is a sequence of reducing spheres  $P = P_0, P_1, \dots, P_n = Q$  so that  $P_{i-1} \cdot P_i = 4, i = 1, \dots, n$ . The proof will be by induction on the length  $n$  of this sequence – the case  $n = 1$  follows from Corollary 3. In particular, there is a word  $\omega$  in the generators  $\alpha, \beta, \gamma, \delta$  so that  $\omega(P_1) = P$ . Apply  $\omega$  to every sphere in the shorter sequence  $P_1, \dots, P_n = Q$  and obtain a sequence  $P = \omega(P_1), \omega(P_2), \dots, \omega(Q) = \omega(h(P))$ . Then by inductive hypothesis,  $\omega h$  is in the group generated by  $\alpha, \beta, \gamma, \delta$ , hence so is  $h$ .  $\square$

There are several natural choices for  $\delta$ . For example, if we think of  $V$  as a ball with two 1-handles attached, the two 1-handles separated by the reducing sphere  $P$ , then a slide of an end of one of the 1-handles over the other around a longitudinal curve will suffice for  $\delta$ . This is the genus two version of Powell's move  $D_\theta$  ([Po, Figure 4]). Another possibility is to choose an order two element for  $\delta$ , an element that is conjugate in  $\mathcal{H}$  to  $\gamma$ : note from Figure 2 that  $Q \cdot \gamma(Q) = 4$ .

A bit more imaginative is the automorphism shown in Figure 10 which is of order three and corresponds to rotating one of the two-simplices of  $\Gamma$  around its center. The figure is meant to evoke a more symmetric version of Figure 2: it depicts a thrice punctured sphere

with three essential arcs, each pair intersecting in two points. Thicken the figure (i.e. cross with an interval). Then the thrice punctured sphere becomes a genus two handlebody  $V$  and each arc becomes a disk. Each disk is the intersection with  $V$  of a reducing sphere, and the three reducing spheres are represented by the corners of a single two-simplex  $\sigma$  in  $\Gamma$ . Rotation of the figure by  $2\pi/3$  along the axis shown cyclically permutes the three arcs, and so cyclically permutes the three reducing spheres. Hence it also rotates the corresponding 2-simplex  $\sigma$  in  $\Gamma$ .

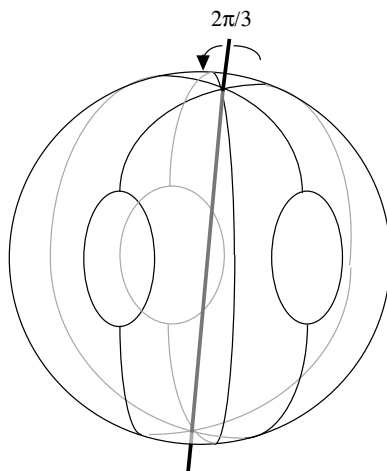


FIGURE 10

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MATHEMATICS DEPARTMENT, UNIVERSITY OF CALIFORNIA, SANTA BARBARA, CA 93106, USA

*E-mail address:* mgscharl@math.ucsb.edu