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THOROUGHLY KNOTTED HOMOLOGY SPHERES

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For H^n a homology n-sphere, consider the problem of classifying locally flat imbeddings $H^n \hookrightarrow S^{n+2}$ up to isotopy. Since any imbedding may be altered by adding knots $S^n \hookrightarrow S^{n+2}$, the classification problem is at least as complex as the isotopy classification of knots. Elsewhere [8] we show that there is a natural correspondence between knot theory and the classification of those imbeddings $H^n \hookrightarrow S^{n+2}$ which satisfy a certain condition on fundamental group. Those imbeddings which do not satisfy the fundamental group condition will be called thoroughly knotted. The goal of the present paper is the construction, for all $n \geqslant 3$, of homology spheres H^n and of PL locally flat imbeddings $H^n \hookrightarrow S^{n+2}$ which are thoroughly knotted. A corollary will be that, for these homology spheres, the codimension two classification problem is measurably more complex than knot theory.

The outline is as follows: In §1 we define thoroughly knotted imbeddings and present, for any homology n-sphere H^n, necessary conditions for $\pi_1(H)$ to be the fundamental group of a thoroughly knotted homology n-sphere. These conditions are shown to be sufficient if $n \geqslant 5$. Also, for $n \geqslant 3$, the conditions are shown to be sufficient to produce a thoroughly knotted imbeddings $H \# H \hookrightarrow S^{n+2}$.

In §2 we review enough of Milnor's K-theory and of Steinberg's results on Chevalley groups to produce in §3 an example of a group (the binary icosahedral group) satisfying the algebraic conditions of §1, and, therefore, providing thoroughly knotted homology n-spheres for all $n \geqslant 3$.

Finally, in §4 we use the same example to show that for $n = 3$ the algebraic conditions of §1 are not sufficient to produce thoroughly knotted imbeddings of H itself, but only of $H \# H$.

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§1. **The algebraic problem.** Suppose a homology n-sphere H^n is locally flatly imbedded in S^{n+2} and let $i\colon H^n \times D^2 \hookrightarrow S^{n+2}$ be an imbedding of a tubular neighborhood. Let $\pi_1(H) = K$ and let G be the commutator subgroup of $\pi_1(S^{n+2} - H)$. Since $H_1(S^{n+2} - H) \simeq Z$, $\pi_1(S^{n+2} - H)$ may be written as a semidirect product $G \oplus_\tau Z$. In other words, there is an automorphism $\tau\colon G \to G$ such that $\pi_1(S^{n+2} - H)$ consists of ordered pairs (g,t), $g \in G$, $t \in Z$ with group multiplication $(g,t)\cdot(g',s) = (g\cdot\tau^t(g'),t + s)$.

Van Kampen's theorem gives the following push-out diagram of fundamental groups, induced by inclusion:

$$\begin{array}{ccc} \pi_1(H \times \partial D^2) \simeq K \oplus Z & \xrightarrow{\varphi_1} & G \oplus_\tau Z \\ \Big\downarrow \varphi_2 & & \Big\downarrow \psi_1 \\ K & \xrightarrow[\psi_2]{} & 1 \end{array}$$

DEFINITION. The imbedding $i\colon H \hookrightarrow S^{n+2}$ is thoroughly knotted if $\varphi_1(K \oplus \{0\}) \neq 0$.

The importance of this phenomenon will be explored in [8].

Note that, since $S^{n+2} - i(H \times \mathring{D}^2)$ is compact, $G \oplus_\tau Z$ is finitely generated (though G need not be). Furthermore, $H_1(G \oplus_\tau A) \simeq H_1(S^{n+2} - H) \simeq Z$ and, by the Hopf theorem [3],

$$H_2(G \oplus_\tau Z) \simeq \frac{H_2(S^{n+2} - H)}{\rho\pi_2(S^{n+2} - H)} = 0,$$

where ρ is the Hurewicz homomorphism.

In sum, if $i\colon H \to S^{n+2}$ is a thoroughly knotted imbedding we have

(a) a push-out diagram
$$\begin{array}{ccc} K \oplus Z & \xrightarrow{\varphi_1} & G \oplus_\tau Z \\ \Big\downarrow \varphi_2 & & \Big\downarrow \psi_1 \\ K & \xrightarrow[\psi_2]{} & 1 \end{array}$$

such that

(b) $K \simeq \pi_1(H)$

(c) $\varphi_1(K \oplus \{0\}) \neq 0$

(d) $G \oplus_T Z$ is finitely presented,

$$H_1(G \oplus_T Z) = Z, \; H_2(G \oplus_T Z) = 0.$$

(e) φ_2 is projection and φ_1 commutes with projection to Z.

The following is a partial converse. Let H_- denote H with an open n-disk removed.

1.1 PROPOSITION. *Suppose there is a push-out diagram satisfying (a) - (e). Then there is a compact contractible (n+2)-manifold N with $\partial N \simeq \partial(H_- \times D^2)$ such that*

(i) *the natural inclusion $\partial H_- \times \{0\} \hookrightarrow \partial N$ extends to an imbedding $(H_-, \partial H_-) \times D^2 \hookrightarrow N, \partial N$*

(ii) *the diagram induced by inclusions*

$$\begin{array}{ccc} \pi_1(H_- \times \partial D^2) & \longrightarrow & \pi_1(N - (H_- \times \{0\})) \\ \downarrow & & \downarrow \\ \pi_1(H_- \times D^2) & \longrightarrow & \pi_1(N) \end{array}$$

is the given push-out diagram.

We begin the proof with a preliminary

1.2 LEMMA. *There is a closed m-manifold M, $m \geqslant 5$, such that $\pi_1(M) \simeq G \oplus_T Z$ and $H_i(M) \simeq H_i(S^1 \times S^{m-1})$.*

PROOF. Let $\{x_1,...,x_s; r_1,...,r_t\}$ be a presentation for $G \oplus_T Z$. Identify in the natural way $\pi_1(\#_s(S^1 \times S^4))$ with the free group generated by $x_1,...,x_s$. Perform surgery on circles α_i: $S^1 \hookrightarrow \#_s(S^1 \times S^4)$ $i = 1,...,t$ representing $r_1,...,r_t$. By standard arguments, framings may be chosen for neighborhoods of the $\alpha_i(S^1)$ so that the resulting manifold M' is almost parallelizable. From the Mayer-Vietoris sequence, $H_2(M')$ is free, and from VanKampen's theorem $\pi_1(M') = G \oplus_T Z$.

According to Hopf [3], $0 = H_2(G \oplus_T Z) = H_2(M')/\rho\pi_2(M')$. Thus a basis for $H_2(M')$ is representable by a finite collection of imbeddable spheres $\gamma_i(S^2)$. Since M' is almost parallelizable, each $\gamma_i(S^2)$ has trivial normal bundle. Now perform surgery on each $\gamma_i(S^2)$ to obtain a closed manifold M″ with $\pi_1(M'') \simeq G \oplus_\tau Z$ and $H_2(M'') = 0$. By Poincare duality $H_3(M'') = H_2(M'') = 0$ and $H_4(M'') = H_1(M'') = H_1(G \oplus_\tau Z) = Z$. This completes the construction if m = 5.

If m > 5 remove an open tubular neighborhood in M″ representing a generator of $H_1(M'')$, and take the cross-product of this manifold with D^{m-5}. The resultant manifold is a homology $S^1 \times D^{m-1}$ with fundamental group $G \oplus_\tau Z$ and boundary = $\partial(S^1 \times D^4 \times D^{m-5}) = S^1 \times S^{m-2}$. Attach $S^1 \times D^{m-1}$ by identifying the boundaries. It is easily verified that the resulting manifold M has $\pi_1(M) \simeq G \oplus_\tau Z$ and $H_*(M) \simeq H_*(S^1 \times S^{m-1})$. This proves 1.2.

PROOF OF 1.1. Take m = n + 2 in 1.2 and consider the parallelizable manifold $(H_- \times S^1 \times I) \# M$ with fundamental group $(K \oplus Z) * (G \oplus_\tau A)$. For $y_1,\ldots,y_q$ generators of some presentation for $K \oplus Z$, perform framed surgery on q circles in $(H_- \times S^1 \times I) \# M$ representing the words $y_i^{-1} \cdot \varphi_1(y_i) \in (K \oplus Z) * (G \oplus_\tau Z)$ and call the resulting parallelizable manifold M′. An easy calculation shows $\pi_1(M') \simeq G \oplus_\tau Z$, $H_2(M') \simeq H_n(M')$ is free of rank q - 1 and $H_i(M') = 0$, $i \neq 1,2,n$. Furthermore, $\partial M' \simeq \partial(H_- \times S^1 \times I)$ and the inclusion $H_- \times S^1 \times \{0\} \to M'$ induces the map φ_1 on fundamental groups.

As in Lemma 1.2, since $H_2(G \oplus_\tau Z) = 0$ it is possible to do surgery on q - 1 2-spheres in M′ to obtain N′ a homology $S^1 \times D^{n-1}$ with $\pi_1(N') \simeq G \oplus_\tau Z$ and $\partial N' \simeq \partial(H_- \times S^1 \times I)$.

Finally, let N be the manifold obtained from N′ by attaching $H_- \times D^2$ by a diffeomorphism $H_- \times \partial D^2 \simeq H_- \times S^1 \times \{0\}$. Then $\partial N = \partial(H_- \times D^2)$, $H_i(N) = 0$, $i \geqslant 1$. Van Kampen's theorem implies that $\pi_1(N)$ is the push-out of the maps φ_1 and φ_2, hence, by hypothesis (a), $\pi_1(N)$ is trivial. Thus the imbedding $H_- \times D^2 \hookrightarrow N$ satisfies the requirements of 1.1.

Before trying to improve 1.1 we note the following corollary, which is the central result of this section.

1.3 COROLLARY. *If there is a push-out diagram satisfying (a) - (e) above, then there is a thoroughly knotted imbedding of the double H # (-H) of H into* S^{n+2}.

PROOF. The double along ∂N of the manifold N of 1.1 is a homotopy n+2-sphere, hence S^{n+2}, and contains the double H # (-H) of H. The complement of a tubular neighborhood of $H \# (-H)$ in S^{n+2} is the double of the manifold N' appearing in the proof of 1.1 along $H\_ \times S^1 \times \{1\} \subset \partial N'$. Hence $\pi_1(S^{n+2}\text{-}(H\#H))$ is the free product of $\pi_1(N')$ with itself amalgamated along $\varphi_1(K \oplus Z) \subset G \oplus_\tau Z \simeq \pi_1(N')$. In particular the inclusion $N' \to S^{n+2}\text{-}(H\#H)$ induces an injection $\pi_1(N') \to \pi_1(S^{n+2}\text{-}(H\#H))$. Since $\varphi_1(K \times \{0\})$ is non-trivial in $\pi_1(N')$, its image in $\pi_1(S^{n+2}\text{-}(H\#H))$ is non-trivial. Hence $H \# H$ is thoroughly knotted.

The strongest sufficiency theorem possible for constructing thoroughly knotted homology spheres would be a version of 1.1 in which we replace $H\_$ by H and N by S^{n+2}. Is such a theorem possible? In §4 a counterexample is given in case $n = 3$, but for $n \geqslant 5$ we prove the following slightly weaker version:

1.4 PROPOSITION. *Suppose there is a push-out diagram satisfying (a) - (e). Then for any $m \geqslant 5$ there is a homology m-sphere H' and an imbedding $H' \times D^2 \hookrightarrow S^{m+2}$ such that the diagram induced by inclusions*

$$\begin{array}{ccc} \pi_1(H' \times \partial D^2) & \longrightarrow & \pi_1(S^{m+2} - (H' \times \{0\})) \\ \downarrow & & \downarrow \\ \pi_1(H' \times D^2) & \longrightarrow & \pi_1(S^{m+2}) \end{array}$$

is the given push-out diagram.

Note that we are free to choose $m \neq n$, yet even when $m = n$ we do not know that $H' = H$.

PROOF OF 1.4. In [5] Kervaire shows that, if a group K is the fundamental group of a homology sphere, then $H_i(K) = 0$ for $i = 1,2$, and, conversely, any finitely presented group K with $H_i(K) = 0$, $i = 1,2$, is the fundamental group of a homology m-sphere for each $m \geqslant 5$. In fact, the latter construction creates the homology m-sphere H' with $\pi_1(H') \simeq K$ by beginning with S^m and successively doing surgery on i-spheres for $i = 0,1,2$. It is not difficult to see from his construction that, if W is the trace of the surgeries (beginning with the 0-handle D^{m+1} with boundary S^m), then W is a homology disk with boundary $\partial W = H$, and the inclusion $H \to W$ induces an isomorphism of fundamental groups.

It will be convenient to view $H' \simeq \partial W$ as $H'\_ \cup_\partial D^m$. Consider the manifold N', constructed as in 1.1, whose boundary is $\partial(H'\_ \times S^1 \times I)$. Attach $W \times S^1$ to $H'\_ \times$

$S^1 \times \{1\} \subset \partial N'$ along $H'_- \times S^1 \subset W \times S^1$. This manifold has boundary $(H'_- \times S^1 \times \{0\}) \cup_\partial (D^m \times S^1) \simeq H' \times S^1$. Standard calculations again show that if $H' \times D^2$ is attached to this $H' \times S^1$, the resulting manifold is S^{m+2} and that, since $i: H_- \hookrightarrow W$ induces an isomorphism on fundamental groups, this imbedding of $H' \times D^2$ into S^{m+2} has a push-out diagram as in (ii) of 1.1 and hence as required to prove 1.3.

§2. **K-theory and Chevalley groups.** It is relatively easy to produce examples of push-out diagrams satisfying all the hypotheses of 1.1 other than $H_2(G \oplus_T Z) = 0$. Unfortunately, this last condition is tedious to verify and difficult to produce. In this section we review the algebra (associated with K-theory) which shows that $H_2(SL(n,p)) = 0$. Here $SL(n,p)$ is the special linear group of $n \times n$ matrices, $n \geqslant 5$, in the field Z_p. In §3 we will choose G to be one of these groups. See [7] [10] for missing proofs.

Kervaire and Steinberg provide the following machinery for constructing groups with trivial first and second homology: Given a perfect group G, there is an extension G_c of G, known as the universal central extension, which is characterized by the existence of a surjection $\gamma: G_c \to G$ whose kernel lies in the center of G_c and which factors uniquely through any other central extension of G.

Milnor constructs G_c as follows [7]: Let F be a free group which maps onto G and let R be the kernel of the map. Since G is assumed perfect, $[F,F]/[F,R]$ also maps onto G, and the kernel, $(R \cap [F,F])/[R,F]$ is clearly central. Milnor provides an elegant proof that $G_c \simeq [F,F]/[F,R]$. Note that the center $(R \cap [F,F])/[R,F]$ of G_c is, according to Hopf [3], precisely $H_2(G;Z)$.

2.1 LEMMA. *For any perfect group G, $H_2(G_c) \simeq H_1(G_c) \simeq 0$. Furthermore, if $H_2(G) = H_1(G) = 0$, then $G_c \simeq G$.*

PROOF. By [7, Theorem 5.3] G_c is perfect and, for $(G_c)_c$ the universal central extension of G_c, the exact sequence

$$1 \longrightarrow H_2(G_c) \longrightarrow (G_c)_c \xrightarrow{\phi} G_c \longrightarrow 1$$

splits. Let $s: G_c \to (G_c)_c$ be the splitting (i.e. $\phi s = 1$). By definition of $(G_c)_c$, there is a *unique* homomorphism $(G_c)_c \xrightarrow{h} (G_c)_c$ such that $\phi \cdot h = \phi$. Since $\phi s \phi = \phi$, $s\phi = 1$. Then ϕ is an isomorphism and so $H_2(G_c) = 0$.

If $H_2(G) \simeq H_1(G) \simeq 0$, then $\ker(G_c \to G)$ is trivial and so $G_c \simeq G$. This completes the proof of 2.1.

REMARK. An (unimportant) corollary of this lemma is that a group G is the fundamental group of a homology m-sphere, $m \geqslant 5$ if and only if G is a finitely presented universal central extension. Unfortunately, the presentation $[F,F]/[F,R]$ of G_c given by Milnor is finitely generated only if it is trivial.

Steinberg has studied a class of groups for which a presentation for the universal extension can be calculated in a different manner. To each of these groups $G(\Sigma,F)$, called Chevalley groups, is associated a field F and a root system Σ of a semi-simple Lie algebra. The most important example to topologists has been the root system of type A_{n-1}. In this case $G(A_{n-1},F)$ is $SL(n,F)$, the group of unimodular $n \times n$ matrices with coefficients in F. For $n \geqslant 5$, its universal central extension $St(n,F) \to SL(n,F)$, generated by the Steinberg symbols, has kernel Milnor's $K_2(F)$.

For F a finite field, $K_2(F) = 0$. Thus we have

2.2 PROPOSITION. (Steinberg). *For F a finite field and $n \geqslant 5$, SL(n,F) is a finite universal central extension.*

§3. **The algebraic construction.** In this section we present a push-out diagram satisfying the hypotheses of 1.1,with H the dodecahedral space. The existence of a thoroughly knotted imbedding of the double of the dodecahedral space (in fact, examples in all dimensions) will then follow from 1.3.

Let $K = \pi_1(H) = SL(2,5)$ and $G = SL(5,5)$. There is a natural injection j: $SL(2,5) \oplus SL(3,5) \to SL(5,5)$ induced by the decomposition $(Z_5 \oplus Z_5) \oplus (Z_5 \oplus Z_5 \oplus Z_5)$. SL(2,5) has one non-trivial normal subgroup, the center $C \simeq Z_2$ with one non-zero element $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. In general, SL(m,n) is simple if m and n - 1 are relatively prime [2]. In particular SL(3,5) and SL(5,5) are simple. Let $\alpha',\beta' \in SL(3,5)$ have non-trivial commutator $[\alpha',\beta']$ and let $\alpha = j(1 \oplus \alpha')$, $\beta = j(1 \oplus \beta')$. For $G = SL(5,5)$, define $\tau\colon G \to G$ by $\tau(g) = \alpha g \alpha^{-1}$. Clearly $\tau | j(SL(2,5) \oplus 1)$ is the identity, so the homomorphism $\varphi_1\colon K \oplus Z \to G \oplus_\tau Z$ given by $\varphi_1(k,t) = (j(k \oplus 1),t) \in G \oplus_\tau Z$ is well-defined and injective.

Let $\varphi_2\colon K \oplus Z \to K$ be the projection, and consider the push-out

$$\begin{tikzcd} & G \oplus_\tau Z \arrow[dr, "\psi_1"] & \\ K \oplus Z \arrow[ur, "\varphi_1"] \arrow[dr, "\varphi_2"'] \arrow[rr, dashed, "\psi_3"] & & X \\ & K \arrow[ur, "\psi_2"'] & \end{tikzcd}$$

3.1 PROPOSITION. *The above push-out satisfies properties (a) - (e) of §1.*

PROOF. (a) For any $t \in Z$, $\varphi_2(1,t) = 1$, so $\psi_1\varphi_1(1,t) = 1$. Thus $\ker(\psi_1)$ contains the normal subgroup generated by $\varphi_1(1,t)$. Then for any $(g,0) \in G \oplus_\tau Z$ and $t \in Z$, $\psi_1(g,0) = \psi_1(\varphi_1(1,t)\cdot(g,0)\cdot\varphi_1(1,-t)) = \psi_1(\tau^t(g),0)$. In particular, for $g = \beta^{-1}$ and $t = 1$, $\psi_1(\beta\alpha\beta^{-1}\alpha^{-1},0) = 1$.

Since $[\beta,\alpha] \neq 1$, $\ker \psi_1 \cap G \subset G \oplus_\tau Z$ is a non-trivial normal subgroup of G. Since G is simple, $\ker \psi_1$ contains G as well as Z. Hence ψ_1 is trivial, so ψ_3 is trivial and, finally, $X = 1$.

(b) $K = SL(2,5) = \pi_1$ (Dodecahedral space).

(c) φ_1 is injective by definition.

(d) Since $G \simeq SL(5,5)$ is finite, $G \oplus_\tau Z$ is finitely presented. Since G is simple, $H_1(G) = 0$ and $H_1(G \oplus_\tau Z) = Z$. To show $H_2(G \oplus_\tau Z) = 0$, let Y be a $K(G \oplus_\tau Z,1)$ space, The homomorphism $\pi_1(Y) \to Z$ yields a covering $K(G,1)$ space $\widetilde{Y}$. Then, since $H_2(G) = 0$, $H_2(\widetilde{Y}) = H_1(\widetilde{Y}) = 0$. Hence $H_2(G \oplus_\tau Z) = H_2(Z) = 0$ [6, Corollary 7.3].

(e) Follows from the definitions.

3.2 COROLLARY. *There are thoroughly knotted homology n-spheres for all $n \geqslant 3$.*

PROOF. By 1.3 it suffices to produce for each $n \geqslant 3$ a homology n-sphere with $\pi_1(H) \simeq SL(2,5)$. For $n = 3$, let H be the dodecahedral space. For $n > 3$, use $\partial(H\_ \times D^{n-2})$.

§4. Algebra ≠ Geometry for n = 3. The purpose of this section is to show that the diagram of 3.1 is a counterexample to 1.4 in the case m = 3. The fundamental tool is

4.1 THEOREM. *Suppose N is a compact orientable 4-manifold with connected boundary M such that $\pi_1(M) \simeq SL(2,5)$ and $\pi_1(N)$ maps onto $SL(5,5)$. Then the composition $\pi_1(M) \to \pi_1(N) \to SL(5,5)$ can not be the natural inclusion $SL(2,5) \hookrightarrow SL(5,5)$.*

First we show how 4.1 produces a counterexample.

Suppose there is a *smooth* imbedding of a homology 3-sphere H in S^5 such that the induced push-out diagram is that of 3.1:

$$\begin{array}{ccc} SL(2,5)\oplus Z & \longrightarrow & SL(5,5)\oplus_T Z \\ \downarrow & & \downarrow \\ SL(2,5) & \longrightarrow & 1 \end{array}$$

Let $i\colon H \times D^2 \hookrightarrow S^5$ be a tubular neighborhood of H.

Consider the infinite cyclic cover X of $S^5 - (H \times \mathring{D}^2)$. X has two ends and boundary $H \times R$. Extend the projection $H \times R \to R$ to a proper map $f\colon X \to R$ and homotope rel boundary to make f transverse to 0 in R. By standard surgery arguments $N = f^{-1}(0)$ may be chosen so that N is connected, and, since G is finitely generated, so that the inclusion $i'_{\#}\colon \pi_1(N) \to G$ is surjective. This contradicts 4.1.

We can eliminate the possibility of such a non-smoothable locally flat imbedding $H \hookrightarrow S^5$ in either of two ways. The first: apply the transversality theory of [9] to produce a homology manifold satisfying the hypotheses of N in 4.1 but not the conclusion, and observe that the proof of 4.1 applied also to N a homology manifold. The second method: add to H a non-smoothable knot $S^3 \hookrightarrow S^5$ with $\pi_1(S^5 - S^3) \simeq Z$, to make H smoothable [1].

We begin the proof of the theorem.

LEMMA 1. *The 3-Sylow subgroups of SL(2,5) and SL(5,5) are, respectively,* Z_3 *and* $Z_3 + Z_3$.

PROOF. A calculation [2, page 491] shows that the highest power of 3 dividing the orders of SL(2,5) and SL(5,5) are 3 and 9 respectively. Thus the 3-Sylow subgroup of SL(2,5) is Z_3 and the inclusion $Z_3 + Z_3 \subset SL(2,5) \oplus SL(2,5) \subset SL(5,5)$ is a 3-Sylow subgroup of SL(5,5).

For V a vector space, let $V \cdot V$ denote the symmetric product of V with itself.

LEMMA 2. *There is a natural isomorphism of* Z_3*-vector spaces* $H^2(Z_3 + Z_3;Z) \cdot H^2(Z_3 + Z_3;Z) \xrightarrow{\varphi} H^4(Z_3 + Z_3;Z)$ *given by* $\varphi(a \cdot b) = a \cup b$.

PROOF. In general

$$H^q(Z_p;Z) \simeq \begin{cases} Z_p & q \text{ even} \\ 0 & q \text{ odd} \end{cases} \qquad [6].$$

It follows from the integral Bockstein sequence that the Bockstein map β:

$H^{2k-1}(Z_3;Z_3) \to H^{2k}(Z_3;Z)$ is also an isomorphism, and from the universal coefficient theorem that the reduction homomorphism r: $H^{2k}(Z_3;Z) \to H^{2k}(Z_3;Z_3)$ is also an isomorphism. Let g generate $H^2(Z_3;Z)$.

Since the 3-skeleton of $K(Z_3,1)$ is a Lens space, hence a manifold, it follows by Poincare duality in the field Z_3 that $\beta^{-1}(g) \cup g$ generates $H^3(Z_3;Z_3)$, so $r\beta(\beta^{-1}(g) \cup rg) = rg \cup rg + \beta^{-1}(g) \cup \beta(rg)$ generates $r(H^4(Z_3;Z))$. Since rg is in the image of $r\beta$, $\beta(rg) = 0$. Hence $r(g \cup g)$ generates $r(H^4(Z_3;Z_3))$ and so $g \cup g$ generates $H^4(Z_3;Z)$. Denote $g \cup g$ by h.

Since $(H^*(Z_3;Z) * H^*(Z_3;Z))^5 = 0$, the Künneth formular provides a natural isomorphism,

$$(H^*(Z_3;Z) \otimes H^*(Z_3;Z))^4 \simeq H^4(Z_3 + Z_3;Z),$$

so $H^4(Z_3 + Z_3;Z)$ is generated by $h \times 1, g \times g, 1 \times g$.

Let p_i: $Z_3 + Z_3 \to Z_3$ be the projections, $i = 1,2$, and $g_i = p_i^*(g)$. Then $H^4(Z_3 + Z_3;Z)$ is generated by $p_1^*(h) = g_1 \cup g_1$, $g_1 \cup g_2$, and $p_2^*(h) = g_2 \cup g_2$.

The Künneth formula also gives a natural isomorphism

$$(H^*(Z_3;Z) \otimes H^*(Z_3;Z))^2 \to H^2(Z_3 + Z_3;Z)$$

sending $g \otimes 1$ to g_1 and $1 \otimes g$ to g_2. Thus $H^2(Z_3 + Z_3;Z) \cdot H^2(Z_3 + Z_3;Z)$ is generated by $g_1 \cdot g_1$, $g_1 \cdot g_2$, and $g_2 \cdot g_2$.

Since φ maps a basis of the Z_3-vector space $H^2(Z_3 + Z_3;Z) \cdot H^2(Z_3 + Z_3;Z)$ to a basis of the Z_3-vector space $H^4(Z_3 + Z_3;Z)$, φ is an isomorphism. Q.E.D.

Let $\gamma = \begin{Bmatrix} 0 & 1 \\ -1 & -1 \end{Bmatrix}$ in SL(2,5). Then $\gamma^3 = 1$, so γ generates a subgroup $Z_3 \subset SL(2,5) \overset{j}{\hookrightarrow} SL(5,5)$.

LEMMA 3. *The map $H^3(SL(5,5);Z_3) \to H^3(Z_3;Z_3)$, induced by inclusion, is surjective.*

PROOF. Let $N \subset SL(5,5)$ be the normalizer of the 3-Sylow subgroup generated by the matrices

$$a = \begin{pmatrix} \gamma & & \\ & & \\ & & I \end{pmatrix} \quad \text{and } b = \begin{pmatrix} I & & \\ & \gamma & \\ & & 1 \end{pmatrix}.$$

Then $Z_3 \subset N$ is generated by a. The map $H^3(SL(5,5);Z_3) \to H^3(N;Z_3)$ is an isomorphism [4]. Thus it suffices to show $H^3(N;Z_3) \to H^3(Z_3;Z_3)$ is surjective.

From the Lyndon spectral sequence we deduce that the 3-primary torsion of $H^4(N;Z)$ is isomorphic to the submodule of $H^4(Z_3 + Z_3;Z)$ fixed by the action (via conjugation) of N on $Z_3 + Z_3$ [6, page 117, 352]. We now examine this action.

Write $x \in N \subset SL(5,5)$ as

$$x = \left(\begin{array}{c|c} X_1 & X_2 \\ \hline \underbrace{X_3}_{2} & \underbrace{X_4}_{3} \end{array}\right)\begin{array}{l}\}2\\ \}3\end{array}$$

Since $x \in N$, $xax^{-1} = a^n b^m$ for some $n,m \in Z_3$. Then $xa = a^n b^m x$ implies $X_2 = \gamma^n X_2$, so $(I - \gamma^n)X_2 = 0$. But if $n \neq 0$ in Z_3 then $(I - \gamma^n) = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$ or $\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$. In either case the two rows of X_2 are equal.

Similarly $xb = a^{n'} b^{m'} x$ for some $n',m' \in Z_3$, so $X_1 = \gamma^{n'} X_1$ and $(I - \gamma^{n'})X_1 = 0$. Then if $n' \neq 0$, X_1 has both rows equal. Since $\det x \neq 0$, one of n or n′ must be zero. Furthermore, since $X_4 = b^m X_4$ and $X_3 = b^{m'} X_3$, it follows in a similar manner that one of m or m′ is zero.

Since conjugation by x is an automorphism of $Z_3 + Z_3$, the matrix $\begin{pmatrix} n & n' \\ m & m' \end{pmatrix}$ must be non-singular. Hence it is one of the following eight matrices

$$\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}$$

The universal coefficient theorem gives an isomorphism

$$H^2(Z_3 + Z_3;Z) \simeq \mathrm{Ext}(H_1(Z_3 + Z_3;Z),Z).$$

Since Ext is contravariant in its first variable, an automorphism of $H_1(Z_3 + Z_3;Z)$ represented by a matrix A, induces an automorphism of $H^2(Z_3 + Z_3;Z)$ represented by A^T on its natural basis ($g \times 1, 1 \times g$). Thus any x in N operates on $H^2(Z_3 + Z_3;Z)$ by a transpose of one of the above matrices, hence by one of the above matrices.

By Lemma 2, the action of x on $H^4(Z_3 + Z_3;Z)$, with respect to the basis given in the proof of that lemma, is given by the symmetric product of one of these eight matrices with itself. There are four possibilities:

$$\begin{pmatrix} 1 & & \\ & \pm 1 & \\ & & 1 \end{pmatrix}, \quad \begin{pmatrix} & & 1 \\ & \pm 1 & \\ 1 & & \end{pmatrix}$$

In any case x fixes $g_1 \cup g_1 + g_2 \cup g_2$, so the image of $H^4(N;Z) \to H^4(Z_3 + Z_3;Z)$ contains $g_1 \cup g_1 + g_2 \cup g_2$. Hence $H^4(N;Z) \xrightarrow{i^*} H^4(Z_3;Z)$ is onto.

Since the 3-primary part of $H^4(N;Z)$ is a submodule of $H^4(Z_3 + Z_3;Z)$, each of its elements has order at most 3. Hence i^* is split by a map s: $H^4(Z_3;Z) \to H^4(N;Z)$.

Consider the commutative diagram with rows the integral Bockstein homomorphism

$$\begin{tikzcd} H^3(N;Z_3) \arrow[r, "\beta"] \arrow[d, "i^*"] & H^4(N;Z) \arrow[r, "3"] \arrow[d, "i^*"] & H^4(N;Z) \arrow[d, "i^*"] \\ H^3(Z_3;Z_3) \arrow[r, "\sim", "\beta"'] & H^4(Z_3;Z) \arrow[r, "3"'] \arrow[u, dashed, bend left, "s"] & H^4(Z_3;Z) \end{tikzcd}$$

Since $H^4(N;Z) \xrightarrow{3} H^4(N;Z)$ is trivial on $s(H^4(Z_3;Z))$, there is an element $x \in H^3(N;Z_3)$ such that $i^*\beta(x)$ generates $H^4(Z_3;Z)$. But since $\beta i^*(x) = i^*\beta(x)$ and β: $H^3(Z_3;Z_3) \to H^4(Z_3;Z)$ is an isomorphism, $i^*(x)$ generates $H^3(Z_3;Z_3)$. Q.E.D.

LEMMA 4. *The homomorphism $H_3(Z_3;Z) \to H_3(SL(5,5);Z)$ is injective.*

PROOF. Consider the commutative diagram obtained from the universal coefficient theorem:

$$\begin{tikzcd} 0 \arrow[d] & 0 \arrow[d] \\ \mathrm{Ext}(H_2(SL(5,5);Z),Z_3) \arrow[r] \arrow[d] & \mathrm{Ext}(H_2(Z_3;Z);Z_3) \arrow[d] \\ H^3(SL(5,5);Z_3) \arrow[r] \arrow[d] & H^3(Z_3;Z_3) \arrow[d] \\ \mathrm{Hom}(H_3(SL(5,5);Z),Z_3) \arrow[r] \arrow[d] & \mathrm{Hom}(H_3(Z_3;Z),Z_3) \arrow[d] \\ 0 & 0 \end{tikzcd}$$

Since $H^3(SL(5,5);Z_3) \to H^3(Z_3;Z_3)$ is onto, $\mathrm{Hom}(H_3(SL(5,5);Z),Z_3) \to \mathrm{Hom}(H_3(Z_3;Z),Z_3)$ is onto. Then $H_3(Z_3;Z) \to H_3(SL(5,5);Z)$ must be injective.

PROOF OF 4.1. In terms of the bordism groups, Theorem 4.1 claims: If α: $M^3 \to K(SL(2,5),1)$ is a map inducing an isomorphism on fundamental groups, then the inclusion $\Omega_3(SL(2,5)) \to \Omega_3(SL(5,5))$ does not map α to zero.

Since $\pi_1(M)$ is finite, $\pi_2(M) = 0$, and so the universal cover of M is a homotopy 3-sphere. Thus $K(SL(2,5),1)$ may be obtained from M by attaching a *single* 4-cell and cells of dimension greater than 4. Hence we may take α to be the inclusion, and further deduce that $H_3(SL(2,5);Z)$ is generated by $\alpha_*[M]$.

Since the composition

$$H_3(Z_3;Z) \to H_3(SL(2,5);Z) \to H_3(SL(5,5);Z)$$

is non-trivial, the generator $\alpha_*[M]$ of $H_3(SL(2,5),Z)$ must be non-trivial in $H_3(SL(5,5);Z)$.

From the commutative diagram

$$\begin{array}{ccc} \alpha \in \Omega_3(SL(2,5)) & \longrightarrow & \Omega_3(SL(5,5)) \\ \downarrow \quad \downarrow & & \downarrow \\ \alpha_*[M] \in H_3(SL(2,5);Z) & \longrightarrow & H_3(SL(5,5);Z) \end{array}$$

it then follows that α must be non-trivial in $\Omega_3(SL(5,5);Z)$.

REFERENCES

1. S. Cappell and J. Shaneson, *Topological knots and knot cobordism*, Topology, 12(1973), 33-40.
2. D. Gorenstein, *Finite groups*, Harper and Row, 1968.
3. H. Hopf, *Fundamentalgruppe und zweite Bettische Gruppe*, Comment. Math. Helv., 14(1941-42), 257-309.
4. D. L. Johnson, *A transfer theorem for the cohomology of a finite group*, Inventiones Math., 7(1969), 174-182.
5. M. Kervaire, *Smooth homology spheres and their fundamental groups*, Trans. Amer. Math. Soc., 144(1969), 67-72.
6. S. MacLane, *Homology*, Springer-Verlag, 1963.
7. J. Milnor, *Introduction to Algebraic K-theory*, Princeton, 1971.
8. M. Scharlemann, *Isotopy and cobordism of homology spheres in spheres*, to appear.
9. M. Scharlemann, *Tranversality theories at dimension four*, Inventiones Math., 33(1976), 1-14.
10. R. Steinberg, *Lectures on Chevalley groups*, (mineographed), Yale, 1967.

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