A PROJECTIVE PLANE IN $\mathbb{R}^4$ WITH THREE CRITICAL POINTS IS STANDARD. STRONGLY INVERTIBLE KNOTS HAVE PROPERTY $P$

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Let $\mathbb{P}$ be the projective plane in $\mathbb{R}^4$ obtained by capping off the boundary of an unknotted Möbius band in $\mathbb{R}^3 \times \{0\}$ with an unknotted disk in $\mathbb{R}^3 \times [0, \infty)$. Here we show that any smoothly imbedded projective plane in $\mathbb{R}^4$ on which some projection $\mathbb{R}^4 \to \mathbb{R}$ has three non-degenerate critical points is isotopic to $\mathbb{P}$. The proof is based on a combinatorial solution to Problem 1.2B of [4]. In particular, if a band is attached to an unknot so that the result is an unknot, then the band is isotopic to the trivial half-twisted band. One consequence is that strongly invertible knots have property $P$ (see [1]). Together with [2], this further implies that pretzel knots (indeed all symmetric knots) have property $P$.

The solution of 1.2B uses the techniques of [5], [6] and [1], with careful distinction made between the two sides of the planar surfaces used in those arguments. Here is the philosophy: It was pointed out in [1] that the techniques of [5] and [6] were inadequate, because a certain type of semi-cycle ([1] Fig. 3) may arise, and yet cannot be used in the reduction process because there is no control over the side of the planar surface on which the interior of the semi-cycle lies. This was easily circumvented in [5], [6] because one of the planar surfaces involved was a punctured sphere. Here the planar surfaces are punctured disks, and it is difficult to detect whether a potential reducing disk is incident to just one side of the disk. In principle, simple bookkeeping should circumvent the problem. Yet there is a crucial step in the argument ([5, 6.31]) at which all distinction between these sides seems hopelessly lost. In 6.9 an extremely weak analogue of [5, 6.31] is recovered, however, and prompts the study of "special paths". These special paths are used, in three successive constructions of "multiflows", to produce a semi-cycle whose interior must lie on a single side of a planar surface, and hence can be used to reduce the complexity of that surface. Roughly, the goal of the three successive stages is to remove from the interior of a multiflow first all sources, then all vertices, then all apexes. During the entire process, care is taken to ensure that at the end, all edges will have the same side on the interior.

§1. THE MAIN THEOREM AND TOPOLOGICAL CONSEQUENCES

1.1 Divide a 3-ball into four quadrants by two 2-disks, $D_v$ and $D_h$, one vertical and one horizontal. Label the quadrants by the points of the compass NE, NW, SW, SE. Let $\mathbb{N}$ be the 3-manifold obtained by attaching two 1-handles to the 3-ball, one connecting SE to NW, the other connecting SW to NE.
1.2 (See Fig. 1.) Let $A_n$ be an imbedded family of simple closed curves in $\partial N$ consisting of circles $a_1, \ldots, a_n$ (labelled south to north) parallel to $\partial D$, together with a circle $a$ going once over each 1-handle. Let $B_n$ be an imbedded family of $n$ simple closed curves in $\partial N$ consisting of circles $b_1, \ldots, b_n$ (labelled west to east) parallel to $\partial D$, together with a circle $\beta$ going once over each 1-handle. Note that there is ambiguity in our definition of $A_n$ and $B_n$ because the number of times $a$ and $\beta$ wrap around each 1-handle as they pass over is left undefined. In particular, if this choice differs for $a$ and $\beta$ by the integers $i_e$ and $i_l$ in the eastern and western 1-handles respectively, there are $|i_e|$ and $|i_l|$ points of intersection in the 1-handles.

**Theorem 1.3.** Suppose that $N$ is imbedded in an oriented 3-manifold $M$ so that some $A_n$ and some $B_n$ bound imbedded planar surfaces $P, Q$ in closure $(M-N)$. Then some $A_0$ and $B_0$ bound imbedded disks $E_P$ and $E_Q$ in closure $(M-N)$ and either:

(a) $E_P$ and $E_Q$ are disjoint and $\mathbb{R}P^3$ is a summand of $M$ or

(b) There is an imbedded disk $D$ in closure $(M-N)$, disjoint from $E_P$ and $E_Q$, such that $\partial D$ is the union along the boundary of two arcs, one crossing a 1-handle once, and the other lying in the boundary of the 3-ball.

**Theorem 1.4.** No 3-manifold obtained by surgery on a strongly invertible knot is simply-connected, i.e. strongly invertible knots have property $P$.

*Proof.* See [1, 1.7].

**Corollary 1.5.** (Seifert, see [3]) Torus knots have property $P$.

**Corollary 1.6.** (Takahashi [7]) 2-bridge knots have property $P$.

*Proof of 1.5 and 1.6.* The torus and 2-bridge knots are strongly invertible.
COROLLARY 1.7. Pretzel knots have property P.

Proof. The n-braid pretzel link $P(a_1, a_2, \ldots, a_n)$ is a knot if either all the $a_i$'s are odd or if exactly one of the $a_i$'s is even. In the former case, $P(a_1, a_2, \ldots, a_n)$ is periodic (with period 2) and so has property P by the recent results of Culler, Gordon, Lueke and Shalen [2]; in the latter $P(a_1, a_2, \ldots, a_n)$ is strongly invertible (see Fig. 2).

Oertel has pointed out that 1.7 extends readily to Montesinos knots (in which the twists in the pretzel knot are replaced by more general rational tangles) and Gordon that it extends in fact to all symmetric knots, though the latter requires the full strength of [2], not just its combinatorics.

1.8 Suppose $\gamma$ is a knot in $S^3$ and a band $b: I \times I \rightarrow S^3$ is attached so that $b^{-1}(\gamma) = I \times \partial I$. Let $\gamma_b$ be the knot obtained by replacing $b(I \times \partial I)$ in $\gamma$ with $b(\partial I \times I)$.

Theorem. Suppose $\gamma$ is unknotted, and so bounds a disk $E_p$ in $S^3$. If $\gamma_b$ is also unknotted, then $\gamma_b$ bounds an unknotted Möbius band in $S^3$ which contains $E_p$.

Proof. Let $N$ be a regular neighborhood of $\gamma \cup b(I \times I)$. By general position (with $b(\{1/2\} \times I)$), $E_p$ can be isotoped so that $b^{-1}(D) = I \times \{a_i\}$ for some finite $\{a_i\} \subset I$. Similarly, by general position with $b(I \times \{1/2\})$, there is a disk $E_Q$ with boundary $\gamma_b$ such that $b^{-1}(D) = \{b_i\}$ for some $\{b_i\} \subset I$. Then it is easy to arrange that the complements $P$ and $Q$ of the interior of $N$ in $E_p$ and $E_Q$ respectively are surfaces satisfying all the hypotheses of 1.3.

Conclusion 1.3(a) does not apply, so 1.3(b) does. Since $m = n = 0$, $E_p \cup b(I \times I)$ is an $I$-bundle on a circle, say with $p$ half-twists. The disk $D$ ensures that the circle is unknotted. Then $\gamma_p$ is a $(2, p)$ torus knot. Since it is unknotted, $p = 1$.

1.9 A surface $M$ in $\mathbb{R}^4$ has $p$ critical points if a projection $p: \mathbb{R}^4 \rightarrow \mathbb{R}$ is a Morse function on $M$ with $p$ non-degenerate critical points.
**Theorem.** If $f$, $g$: $\mathbb{R}P^2 \rightarrow \mathbb{R}^4$ are imbeddings such that $f(\mathbb{R}P^2)$ and $g(\mathbb{R}P^2)$ each have three critical points, then $f$ and $g$ are isotopic.

**Proof.** The three critical points must be a minimum, a maximum, and a single saddle. The proof now follows from 1.8 (cf. [S, 1.3]).

**§2. Preliminaries**

2.1 The points of intersection of $A_m$ and $B_n$ can be labelled as follows: For $1 \leq i \leq m$, $1 \leq s \leq n$ there are 2 points of intersection of $a_i$ with $b_s$, each of which we label $(i, s)$. The curve $\alpha$ intersects the boundary of the 3-ball in two arcs, $\alpha_+$ in the north and $\alpha_-$ in the south. For $1 \leq s \leq n$ label the points of intersection of $b_s$ with $\alpha_+$ and $\alpha_-$ by $(m + 1, s)$ and $(0, s)$ respectively. The curve $\beta$ intersects the boundary of the 3-ball in two arcs, $\beta_+$ in the east and $\beta_-$ in the west. For $1 \leq i \leq m$ similarly label the points of intersection of $a_i$ with $\beta_+$ and $\beta_-$ by $(i + 1, n)$ and $(i, 0)$ respectively. Points of intersection of $\alpha_\pm$ with $\beta_\pm$ are labelled one of $(0, 0)$, $(m + 1, n)$, $(0, n + 1)$, or $(m + 1, n + 1)$ by the obvious convention. Other points of $\alpha \cap \beta$ (those lying in the 1-handles) are unlabelled.

We now proceed just as in [5], [6] and [1]: Suppose $N$ is imbedded in an oriented 3-manifold $M$ in such a way that $A_m$ and some $B_n$ bound planar surfaces $P$ and $Q$ respectively. Assume that we have minimized $m + n$, put $P$ and $Q$ in general position so that intersections consist of arcs and circles, and then (by disk-swapping and isotoping) reduced as much as possible the number of components of intersection of $P$ and $Q$. We have thereby removed all circles in $P \cap Q$ except those essential in both.

Construct as follows semi-oriented graphs $\Gamma_P$ and $\Gamma_Q$ in 2-disks $D_P$ and $D_Q$ with $\partial D_P = \alpha$ and $\partial D_Q = \beta$. Let $D_P$ and $D_Q$ be the disks obtained by filling in disks on the boundary components $a_i$, $1 \leq i \leq m$ and $b_s$, $1 \leq s \leq n$, respectively. Regard each $a_i(b_s)$ as a fat vertex in $\Gamma_P$ ($\Gamma_Q$). Regard the two arcs $\alpha_+$ and $\alpha_-$ ($\beta_+$ and $\beta_-$) as vertices $a_{m+1}$ and $a_0(b_{s+1}$ and $b_0$) in $\Gamma_P$ ($\Gamma_Q$) lying on $\partial D_P$ ($\partial D_Q$). Regard the two arc components of $\alpha(\beta)$ in the 1-handles as two vertices $a_0$ and $a_i(b_s$ and $b_t)$ in $\Gamma_P(\Gamma_Q)$ lying on $\partial D_P(\partial D_Q)$. Finally, regard the arc components of $P \cap Q$ as edges of the graphs $\Gamma_P$ and $\Gamma_Q$. Each end of each edge in $\Gamma_P$ and $\Gamma_Q$ represents a point in $A_m \cap B_n$. The end of an edge in $\Gamma_P(\Gamma_Q)$ not incident to $a_0$ or $a_i(b_s$ or $b_t)$ is assigned the second (first) coordinate of the label above for the corresponding point in $A_m \cap B_n$. In the interior of $D_P(D_Q)$ are $m (n)$ vertices of $\Gamma_P(\Gamma_Q)$, each of valence $2n + 2(2m + 2)$. The labels around the vertices are as shown in Fig. 3. If the first or last points of intersection of $\alpha_\pm$ with $B_n$ are with $\beta_\pm$, push the points into the 1-handles, removing labels 0 and $n + 1$ at $a_0$ and $a_{m+1}$ in $\Gamma_P$ (and labels 0 or $m + 1$ at $b_0$ and $b_{n+1}$ in $\Gamma_Q$). This ensures that $a_0$ and $a_{m+1}$ ($b_0$ and $b_{n+1}$) each have valence $n(m)$.

Orient edges from higher labels to lower; those edges running between identical labels are called level, those edges with an end on $a_0$ or $a_i(b_s$ or $b_t)$ are not oriented and are called bad edges. A simple orientation argument shows that, since all intersections of $\alpha$ and $\beta$ in a 1-handle must have the same sign, no bad edge is a loop. Define a circuit $\gamma$ in $\Gamma_P(\Gamma_Q)$ to be a closed (not necessarily imbedded) path such that for a regular neighborhood $\eta(\gamma)$ of $\gamma$ there is a boundary component $\gamma'$ of $\eta(\gamma)$ such that $\gamma'$ is homotopic to $\gamma$ in $\eta(\gamma)$ and bounds a single disk component of $D_P - \eta(\gamma)(D_Q - \eta(\gamma))$ called the interior of $\gamma$. This is a slight generalization of the definition used in [5], [6] and [1] which insists $\gamma$ be imbedded. Here $\gamma$ is obtained from the simple closed curve by pinching in the exterior of $\gamma'$. Further define interior vertex, chord, spoke, loop, base of a loop, cycle, semi-cycle, label sequence, interior label, sink, and source as in [5] (using this slightly generalized notion of circuit). A good circuit, cycle, or semi-cycle in $\Gamma_P(\Gamma_Q)$ is one which does not contain a vertex $a_0$ or $a_i(b_s$ or $b_t)$. In particular, all circuits lying in the interior of a good circuit are good.
§3. PRELIMINARY TOPOLOGICAL ARGUMENTS

**Lemma 3.1.** Theorem 1.3 is true if \( m = 0 \) or \( n = 0 \).

*Proof.* With no loss of generality assume \( n = 0 \) so \( Q \) is a disk, in which \( P \cap Q \) appears as a collection of arcs. If \( P \cap Q \) is empty, then \( m = n = 0 \) so \( P \) is also a disk and 1.3(a) applies. Otherwise, choose an outermost arc \( \gamma \) in \( Q \). There are four possibilities.

(i) The ends of \( \gamma \) are labelled \( i \) and \( i + 1 \) lying both on \( b_0 \) or both on \( b_{n+1} \). Then \( m \) can be reduced by two using the outermost disk cut off from \( Q \) by \( \gamma \).

(ii) One end of \( \gamma \) lies on \( b_0 \) or \( b_n \) and the other is labelled \( i \) or \( m \). Then \( m \) can be reduced by one using the outermost disk cut off from \( Q \) by \( \gamma \).

(iii) One end of \( \gamma \) is in \( b_0 \) and one in \( b_n \). Then \( m = 0 \), so \( P \) is also a disk. Then the union along \( \gamma \) of the outermost disk of \( Q \) cut off by \( \gamma \) and each of the components into which \( \gamma \) divides the disk \( P \) are the 2-disks satisfying the conclusion of 1.3(b).

The only remaining possibility is:

(iv) The ends of \( \gamma \) are labelled \( 1 \) and \( m \), with one end in \( b_0 \) and one in \( b_{n+1} \). Then the arc of \( \beta \) cut off by the ends of \( \gamma \) must pass once over a one-handle of \( N \). Thus there are exactly two of these outermost arcs, and between them all arcs of \( P \cap Q \) run parallel, with ends labelled \( i \) and \( m - i + 1 \), for \( 1 \leq i \leq m \). (See Fig. 4.) This is impossible if \( m \) is odd (apply [5, 4.2] to \( i = (m + 1)/2 \)) and allows a reduction of 2 in \( m \) if \( m \) is even. The reduction uses a component of \( Q - P \) to do surgery on the union of \( P \) and the annulus on \( \partial N \) between \( a_{m/2} \) and \( a_{m/2+1} \). (See [5, 4.7, case 1]).

Henceforth we therefore assume \( m \neq 0 \neq n \).

**Lemma 3.2.** \( \Gamma_p \) and \( \Gamma_Q \) have no good level loops.

*Proof.* A parity argument as in [5, 4.4].

**Lemma 3.3.** A good cycle in \( \Gamma_p \) or \( \Gamma_Q \) has interior vertices.

*Proof.* See [5, 4.8].

§4. ELEMENTARY COMBINATORICS

**Lemma 4.1.** There is a good cycle in \( \Gamma_p \) or in \( \Gamma_Q \).
Proof. We will assume there are no good cycles and derive a contradiction. First note that if there is any level edge in $\Gamma_p$ with label $s$, say, then the corresponding edge in $\Gamma_q$ is a (good) loop based at $b_s$. Hence we can assume that there are no level edges in either graph.

Call a bad edge in $\Gamma_p$ ($\Gamma_q$) which has label $l$ on one end and other end in $a_s$ or $a_e$ ($b_s$ or $b_e$) a bad $l$-edge. If there are any bad 0-edges or $(m+1)$-edges in $\Gamma_q$ or bad 0-edges or $(n+1)$-edges in $\Gamma_p$, the proof concludes as in [1, 4.1, case ii]. We henceforth assume that there are no bad 0-edges or $(m+1)$-edges in $\Gamma_q$, nor bad 0-edges or $(n+1)$-edges in $\Gamma_p$.

Remove from consideration all bad edges in $\Gamma_q$ to get a new graph $\Lambda$ on the 2 disk $D_2$. Since there are no level edges and each vertex $b_s$, $1 \leq s \leq n$, has a label 0 and a label $m+1$ remaining in $\Lambda$, none of these can be a source or a sink in $\Lambda$. Thus either there is a good cycle, or one of $b_0$ and $b_{n+1}$ is a source (say $b_{n+1}$) and one is a sink (say $b_0$). Double $D_2$ along its boundary to produce a 2-sphere $\Sigma$ containing the double $2\Lambda$ of $\Lambda$. Now proceed as in [6, 6.1]. Since there are no level edges, we can extend the orientations of the edges to a vector field in the neighborhood of the 1-skeleton of $\Lambda$, which has a singularity of index 0 at each vertex $b_s$, $1 \leq s \leq n$, and index 1 at $b_0$ and $b_{n+1}$. The closure $C$ of a component of $\Sigma - \Lambda$ has double along its boundary a surface $T$. The vector field naturally induces a vector field near $\partial T$ whose only singularities (say there are $p$ of them) are sources and sinks. This vector field can be extended to all of $T$, but must, by the Poincaré–Hopf index theorem, have in each copy of $C$ singularities whose indices sum to $\chi(C) - p/2$.

If there is a component $C$ for which there are no singularities in $\partial C$ (i.e. $p = 0$) then $\partial C$ is a cycle, so $\partial C$ cannot come from doubling an arc in $\Lambda$. Hence $C$ is contained in $\Lambda$, and so is a good cycle.

If $p > 0$ on each component $C$ of $\Sigma - \Lambda$, then always $\chi(C) - p/2 \leq 0$. In this case we have constructed on $\Sigma$ a vector field for which every singularity has non-positive index, except $b_0$ and $b_{n+1}$, where the index is 1. Since $\chi(\Sigma) = 2$, we conclude that all inequalities must be equalities and, in particular, $\chi(C) - p/2 = 0$ for every component $C$ of $\Sigma - \Lambda$ and so every component $C$ is a disk and always $p = 2$.

Consider the component $E$ of $D_2 - \Lambda$ which contains the segment of $\partial D_2$ lying between the label 1 in $b_{n+1}$ and the label $m$ in $b_0$ (see Fig. 5). From above, the double of $E$ (along that segment) is a disk $C$, so $E$ is also a disk. Since the vector field defined above has only two singularities along $\partial C$ (in fact a source at $b_{n+1}$ and a sink at $b_0$) the arc $\gamma$ of $\partial E$ lying in $\Lambda$ is an oriented path with tail labelled 1 in $b_{n+1}$ and head labelled $m$ in $b_0$. (All labels of $b_0$ and $b_{n+1}$ belong to edges in $\Lambda$, since there are no bad 0-edges or $(n+1)$-edges in $\Gamma_p$.)

Let $e$ be the (bad) edge in $\Gamma_q$ whose end $\ast$ in $b_s$ is nearest the label 1 in $b_{n+1}$. (Such a bad edge must exist, for otherwise the heads and tails in $\gamma$ would be adjacent, and the labels would be respectively always 1 and always 0.) The other end of $e$ must lie in a vertex $b_s$ lying on the oriented arc $\gamma$, since there are no loops based at $b_s$. Finally, let $F$ be the sub-disk
of $E$, cut off by $p$, whose boundary contains the initial segment of $y$. Then $F$ has no edges or vertices of $\Gamma_Q$ in its interior. Now * represents the point of intersection in the 1-handles of $\alpha$ with $\beta$ located nearest the SE corner of the 3-ball. Push that point of intersection through the SE corner back onto the 3-ball so that it is in $\beta_+ \cap \alpha_-$. The effect is to change * to an end in $\beta_+$ labelled 0, and to change $\partial F$ into a cycle. Then the disk $F$ can be used as in [5, 4.8] to reduce $m$ by one.

**Proposition 4.2.** $\Gamma_F$ and $\Gamma_Q$ each contain good unicyles, and sinks or sources.

*Proof.* See [5, 5.2].

**Proposition 4.3.** Neither $\Gamma_F$ nor $\Gamma_Q$ contains both a source and a sink.

*Proof.* See [5, 5.4].

**Convention 4.4.** By renumbering, if necessary, we assume that both $\Gamma_F$ and $\Gamma_Q$ contain sources but no sinks.

§5. MORE TOPOLOGY: ORIENTATIONS AND NORMAL DIRECTIONS

Picture the 1-handles of $N$ as being attached at points on the boundary of a disk $D_{\perp}$ in the 3-ball which is perpendicular both to $D_x$ and $D_y$. The disk $D_{\perp}$ divides the boundary of the 3-ball from which $N$ is constructed into two hemispheres and $\partial D_{\perp} \cap \alpha = \alpha_{\pm}$, $\partial D_{\perp} \cap \beta = \beta_{\pm}$.

Let $\xi$ and $\zeta$ to be unit normal vector field to $P$ and $Q$ respectively in $M - N$, chosen so that on $\alpha_-$ and $\beta_-$, $\xi$ and $\zeta$ both point into the same hemisphere into which $D_{\perp}$ divides the boundary of the 3-ball. We call the hemisphere the front of the 3-ball, and the other hemisphere (into which $\xi$ and $\zeta$ point along $\alpha_+$ and $\beta_+$) the back. The disk $D_{\perp}$ also divides each simple closed curve $a_t$ and $b_s (t, s \geq 1)$ into two arcs, called the front and back side of the curve, depending on the hemisphere that contains it. Our convention will be to picture $\alpha_\pm$ and $\beta_\pm$ as being pushed slightly into the front face of the 3-ball, allowing us to regard the vertices $a_0$, $a_{m+1}$, $b_0$, and $b_{n+1}$ as always representing arcs on the front face. So, for example, the front face of $a_0$ is in fact all of $a_0$.

It will be convenient to have a shorthand picture which incorporates all the labelling of vertices $a_i$, $1 \leq i \leq m$, and of vertices $b_s$, $1 \leq s \leq n$, and also indicates their front face. The picture is shown in Fig. 6. The shaded region indicates the front, and the heavy dot represents the label $n+1$ in $a_i$ or the label $m+1$ in $b_s$. 

![Fig. 5.](image-url)
5.1 The components \( a_i \) and \( a_m \) together bound a cylinder \( C \) in \( \partial N \). Let \( \eta \) be the tangent unit vector field on \( C \) which points due south (always from \( a_{k+1} \) to \( a_k \)). Then for each \( 1 \leq i \leq m \) there is an \( e_i = \pm \) such that \( \xi = e_i \eta \) on \( a_i \). Define \( e_0 = e_{m+1} = - \). For \( i, j \neq 0 \), \( a_i \) and \( a_j \) are parallel if and only if \( e_i = e_j \). Otherwise they are anti-parallel. Similarly define a sign \( \delta \) for each \( 0 \leq s \leq n + 1 \) by comparing \( \xi \) to a westward pointing vector field \( \nu \) at each \( b_s \), \( 1 \leq s \leq n \) and setting \( \delta_0 = \delta_{m+1} = - \).

5.2 Let \( e \) be an edge in \( \Gamma_Q \) with ends labelled \( i \) and \( j \) at vertices \( b_s \) and \( t_t \), respectively. The edge is said to be \textit{synchronous} if either

(a) \( 1 \leq i, j \leq m \) and the labels \( i + 1 \) at \( b_s \) and \( j + 1 \) at \( b_t \) which are adjacent to \( e \) lie on the same side of \( e \)

(b) \( i = 0 \) (\( m + 1 \)), \( 1 \leq j \leq m \) and the label \( j + 1 \) at \( b_t \) adjacent to \( e \) lies on the same (opposite) side of \( e \) as the front of \( b_t \)

(c) \( i = 0, \) \( j = m + 1 \) (in which case the front of \( b_t \) and the front of \( b_s \) lie on opposite sides of \( e \))

(d) \( i = 0, j = 0 \) (in which case the front of \( b_t \) and the front of \( b_s \) lie on the same side of \( e \)).

Otherwise the edge is \textit{asynchronous}. Define synchronous and asynchronous edges in \( \Gamma_P \) similarly. (See Fig. 7.)

Following is a generalization of [S, 4.21 (all level edges are synchronous).

**Lemma 5.3.** Let \( e \) be an edge in \( \Gamma_Q \) (\( \Gamma_P \)) with ends labelled \( i \) and \( j \) (\( s \) and \( t \)). Then \( e \) is synchronous if and only if \( e_i = e_j (\delta_s = \delta_t) \).

**Proof.** For \( 1 \leq i, j \leq m \), \( e_i = e_j \) if and only if \( \eta \) and \( \xi \) agree at both ends of \( e \), or disagree at both ends of \( e \). The argument for \( i, j = 0 \) or \( m + 1 \) is similar.

5.4 Let \( \gamma \) be a good cycle in \( \Gamma_P (\Gamma_Q) \) such that all edges are asynchronous, at most one of \( 0 \) \( n + 1 \) \((m + 1)\) is an edge label, and neither is an interior label. Let \( v_1, \ldots, v_k \) be the vertices of \( \gamma \) and \( \eta_i \) and \( \tau_i \) denote the labels of the head and tail of \( \gamma \) at \( v_i \), respectively.

**Lemma.** For each \( l \) and interior label \( l \) of \( \gamma \) at \( v_l \), \( \eta_1 < l < \tau_1 \).

**Proof.** Since there are no interior labels \( 0 \) or \( n + 1 (m + 1) \) it follows from Fig. 2 that all interior labels of \( \gamma \) at \( v_i \) lie between \( \eta_i \) and \( \tau_i \). If there are no edge labels \( 0 \) or \( n + 1 (m + 1) \) then since the edges are asynchronous either always \( \eta_i < \tau_i \) or always \( \eta_i > \tau_i \). Since \( \gamma \) is a cycle, all edges are oriented so \( \eta_i > \tau_i \) and \( \eta_i < \tau_k \). But clearly we cannot have \( \eta_1 > \tau_1 > \eta_2 > \ldots > \tau_k > \eta_1 \).
If there is an edge label 0, but not \(n+1(m+1)\) then apply the same argument between successive appearances of 0. Similarly if \(n+1(m+1)\) is an edge label, but 0 is not.

**Lemma 5.5.** Suppose \(e\) is an edge in \(\Gamma_Q\) with one end on the front side of a vertex \(b_i\) and the other on the front side of a vertex \(b_j\). Then \(e\) is asynchronous if and only if \(\delta_i = \delta_j\).

Similarly for an edge in \(\Gamma_P\).

**Proof.** Let \(i\) be the label of \(e\) on \(b_i\) and \(j\) the label on \(b_j\). The normal bundle to \(e\) in \(\Gamma_Q\) and the normal bundle to the corresponding edge in \(\Gamma_P\) together give a normal bundle in \(M-N\) for the arc of intersection of \(P\) with \(Q\) corresponding to \(e\). Since this arc must join points of \(\partial P \cap \partial Q\) of opposite sign, and the ordered pair of vector fields \(\eta, \nu\) used in the definition of \(\varepsilon_i\) and \(\delta_j\), everywhere defines the same orientation on the front face, \(\varepsilon_i \delta_j = \varepsilon_j \delta_i\). The result follows from 5.3.

5.6 Without loss, choose the front side (hence \(\xi\) and \(\zeta\)) so that \(E, = +\). **Warning:** Since \(\delta_1\) may not be +, the roles of \(\Gamma_P\) and \(\Gamma_Q\) may no longer easily be reversed in the statements of theorems.

5.7 Following is a generalization of [5, 4.8].

**Lemma.** A good semi-cycle in \(\Gamma_Q\) has either

(i) a level spoke labelled 0
(ii) a level edge labelled 0 with its back side toward the interior
(iii) an edge or interior label \(m+1\) or
(iv) an interior vertex.

**Proof.** If not, then within it is a semi-cycle \(\gamma\) with neither interior vertices nor interior edges, with all edges labelled \(l\) and \(l+1\), for some \(l < m\), and with the front side of all its 0-labelled level edges toward the interior [5, 4.7]. If \(l \neq 0\) then proceed as in [5, 4.7] to reduce \(m\) by two. If \(l = 0\) and the interior of \(\gamma\) is incident only to the front side of its vertices, or if \(\gamma\) is a cycle then proceed as in [5, 4.8] to reduce \(m\) by one.

So suppose there is at least one level edge of \(\gamma\) labelled 0 and that the interior of \(\gamma\) is incident to the back side of at least one vertex. Let \(b_i\) be the first such vertex in \(\gamma\) following such a level edge. Let \(e\) be the edge preceding \(b_i\) and \(f\) the edge succeeding \(b_i\) in \(\gamma\). (See Fig. 8.) Since the interior of \(\gamma\) lies on the front side of all level edges labelled 0, neither \(e\) nor \(f\) can be such an edge. Hence \(e\) cannot be level with label 1 either, for the oriented edge \(f\) cannot have its head at \(b_i\). Thus \(e\) is oriented with its head at \(b_i\), labelled 0. Our choice of \(b_i\) guarantees that the tail of \(e\) is labelled 1 at the preceding vertex \(b_i\). But then \(e\) is synchronous (5.1(b)), yet \(\varepsilon_i \neq \varepsilon_0\) (5.6), contradicting 5.3.
Fig. 8.

§6. SPECIAL PATHS IN $\Gamma_Q$

DEFINITION 6.1. A good cycle in $\Gamma_p$ is coherent if each of its vertices has subscript smaller than that of each vertex in its interior. A coherent loop based at $a_i$ is a lobe if the interior of the loop is incident only to the front side of $a_i$ (See Fig. 9). A level edge in $\Gamma_Q$ is coherent (lobal) if the corresponding loop in $\Gamma_p$ is coherent (a lobe). Note that any level edge in $\Gamma_Q$ labelled 0 is lobar.

6.2 Since $\Gamma_Q$ contains sources (4.4), any innermost good loop in $\Gamma_p$ is coherent [5, 5.4]. Let $r$ be the highest subscript of all vertices in $\Gamma_p$ which are contained in the interiors of innermost coherent cycles.

LEMMA. (a) Any vertex of a coherent cycle in $\Gamma_p$ has subscript less than $r$.

(b) Any edge incident to $a_i$ is incident to some $a_i$ with $i < r$.

(c) In $\Gamma_Q$ any label $r$ is the tail of an oriented edge.

(d) A level edge in $\Gamma_Q$ with label greater than $r$ is not coherent.

Proof: (a) Contained within the coherent cycle is an innermost one, hence a vertex with subscript $\leq r$.

(b) If the edge were a loop, it would contain an innermost good loop, hence a yet further in coherent cycle.

(c) Follows from b.

(d) The level edge corresponds to a loop in $\Gamma_p$, containing an innermost good loop, hence an innermost coherent cycle, hence a vertex with subscript $\leq r$.

DEFINITION 6.3. The edge $f$ is larger than the edge $e$ in $\Gamma_Q$ if

(i) $e$ and $f$ are incident to the front of the same vertex $b_s$ at labels $i$ and $j$ respectively, with $i < j < r$.

(ii) neither $e$ nor $f$ is oriented with its tail at $b_s$.

(iii) if $e$ or $f$ is level, it is coherent and

(iv) the label of the end of $f$ not incident to $b_s$ is also greater than the label of the corresponding end of $e$. If $e$ or $f$ is lobar, the end of $f$ not incident to $b_s$ need only be $\geq$ the corresponding end of $e$. 

DEFINITION 6.4. A label \( i \geq 1 \) is special if any oriented edge in \( \Gamma_Q \) with tail labelled \( i \) is asynchronous.

DEFINITION 6.5. A special path is a possibly infinite, possibly non-imbedded path \( \mu \) in \( \Gamma_Q \) through oriented edges \( e_1, e_2, e_3, \ldots \) with tails labelled \( \tau_1, \tau_2, \tau_3, \ldots \) respectively, such that for all \( i \geq 1 \):

(a) \( \tau_i \) is special
(b) \( 0 < \tau_i < \tau_{i+1} < \tau \)
(c) each end of each \( e_i \) lies on the front side of the vertex \( b_i \) to which it is incident.
(d) there is a larger edge than \( e_k \) incident to the vertex \( v \) at the head of \( e_k \) if and only if \( e_k \) is the last edge in the path. In this case the vertex \( v \) is called the terminus of \( \mu \), such a larger edge also incident to \( v \) is called a terminator of \( \mu \), and \( e_k \) is the final edge of \( \mu \).

DEFINITION 6.6. A special cycle is a good cycle in \( \Gamma_Q \) of oriented edges \( e_1, \ldots, e_k \) such that the infinite path \( e_1', e_2', e_3', \ldots \) defined by \( e_{(p+k+1)}' = e_p(p \geq 0, 1 \leq i \leq k) \) is a special path.

DEFINITION 6.7. An edge in \( \Gamma_Q \) is special if it belongs to some special path. An edge \( e \) with tail at a vertex \( v \) is extremal if it is special and, of all special edges with tail at \( v \), the tail of \( e \) has highest label. An extremal path (cycle) is a special path (cycle) in which every edge is extremal. See Fig. 10.

LEMMA 6.8. (1) If any initial segment of a special path is deleted, the path is still special.
(2) Suppose \( e_1, e_2, e_3, \ldots \) is a special path \( \mu \). Then either an edge \( e_k \) is the final edge or there is a special path \( e_1', e_2', e_3', \ldots \) such that \( e_i' = e_i \) for \( i < k \) and \( e_{k+1}' \) is extremal.
(3) If a vertex is the tail of a special edge, then it is the initial vertex of some extremal path.

Proof. (1) Follows from the definition 6.5.
(2) Let \( v \) be the vertex at the head of \( e_k \). Either \( e_k \) is the final edge of \( \mu \) or a special edge \( e_{k+1} \) has tail at \( v \). In the latter case, let \( e_{k+1}' \) be that special edge with tail at \( v \) whose label \( \tau_{k+1} \) there is highest. Then \( e_{k+1}' \) is extremal and \( \tau_{k+1} \) is no smaller than \( \tau_{k+1} \), the label of the tail of \( e_{k+1} \). Since \( e_{k+1}' \) is special, it follows from (1) that it is the first edge of a special path \( e_{k+1}', e_{k+2}', e_{k+3}', \ldots \). Then \( e_1, \ldots, e_k, e_{k+1}', e_{k+2}', \ldots \) satisfies (a), (c) and (d) of 6.5 trivially and satisfies (b) since \( \tau_{k+1} \geq \tau_{k+1} \).
(3) Since the vertex is the tail of a special edge, it is the tail of an extremal edge \( e_1 \) which, by (1) is the initial segment of a special path. Suppose inductively that the initial segment \( e_1, \ldots, e_k \) of some special path \( \mu \) beginning with \( e_1 \) has all of its edges extremal. If \( e_k \) is the final edge of \( \mu \) we are done. Otherwise there is a special edge, hence an extremal edge \( e_{k+1} \).
with its tail at the head of $e_k$. By (2) $e_1, \ldots, e_k, e_{k+1}$ is the initial segment of some special path. Continue until either reaching a final edge and hence an entire extremal path or some vertex is encountered for the second time, forming a cycle. Then the path obtained by continuing around the cycle \textit{ad infinitum} is also extremal.

**Lemma 6.9.** Suppose $e$ is a lobal edge in $\Gamma_0$ incident to vertices $b_s$ and $b_t$. Then, either there is a larger edge incident to the front of $b_s$ or $b_t$, or there are disjoint special paths which originate at $b_s$ and $b_t$.

**Proof.** Suppose there is no larger edge incident to the front of $b_s$ or $b_t$. With no loss of generality $s > t$.

Define an $(s-t)$ cycle $\gamma$ (in $\Gamma_p$) to be a coherent cycle in which

(a) all the interior labels of $\gamma$ lie between $s$ and $t$.
(b) every edge is asynchronous
(c) the interior of $\gamma$ is incident only to the front side of its vertices.

Since $e$ (regarded here as a loop in $\Gamma_p$, based at a vertex $a_i$) is a lobe it is also an $(s-t)$ cycle.

**Claim.** Suppose $\gamma$ is an $(s-t)$ cycle in the interior of the loop $e$, and $t$ is the smallest subscript of all vertices in the interior of $\gamma$. Then the edges $f_s$ and $f_t$ incident to the front of $a_i$ at the labels $s$ and $t$ respectively divide $\gamma$ into two circuits, one of which is a coherent cycle $\gamma'$. Moreover, if $e_s = e_j$ for all subscripts $j$ of vertices of $\gamma$, then $\gamma'$ is an $(s-t)$ cycle.

**Proof of claim.** Since $e$ is coherent, any vertex in $\gamma$ or its interior has subscript no smaller than $l$. In particular, $f_s$ and $f_t$, when viewed as edges in $\Gamma_p$ satisfy (i), (iii) and (iv) of 6.3 but by hypothesis are not larger than $e$, so cannot satisfy (ii). Thus the ends of $f_s$ and $f_t$ not incident to $a_i$ are incident to vertices $a_j$ with $j < i$, hence vertices of $\gamma$. (See Fig. 11).

Since all interior labels of $\gamma$ lie between $s$ and $t$, $f_s$ has its head at $a_i$ and $f_t$ its tail. Thus $f_s$ and $f_t$ divide $\gamma$ into two circuits, one of which is a coherent cycle. If, furthermore, $e_i = e_j$ for all subscripts $j$ of vertices of $\gamma$, then by 5.5 every edge of $\gamma$, as well as $f_s$ and $f_t$, are asynchronous. It follows from 5.4 that the head of $f_s$ has label lower than the tail of the next edge of $\gamma$, so, by 5.2, the interior of $\gamma'$ is incident also to $a_i$ only on its front face, between the labels $s$ and $t$. Hence $\gamma'$ is an $(s-t)$ cycle.

**Proof of Lemma 6.9 (Continued).** Since any good cycle (hence $(s-t)$ cycle) must contain interior vertices [3.3], it follows from the claim (used repeatedly) that there must be some
Fig. 11.

(s - t) cycle γ within the loop e in \( \Gamma_p \) for which the interior vertex with smallest label, \( a_i \), has \( e_i \not= e_j \) for every \( a_j \) in γ. Then the label \( i \) is special (5.3 and 6.4). Let \( s_0 = s \) and \( e_1 \) be the edge in \( \Gamma_p \) incident to the label \( s_0 \) on the front of \( a_i \). (The edge \( e_1 \) was called \( f_i \) in the above proof of claim.) Let \( s_1 \) be the label of the other end of \( e_1 \) which, as in the claim, lies on a vertex of γ (so \( t < s_1 < s \)). Let \( e_2 \) be the edge incident to the front of \( a_i \) at label \( s_1 \). If the other end of \( e_2 \) is incident to an interior label of γ, denote it by \( s_2 \) (\( t < s_2 < s \)) and let \( e_3 \) be the edge incident to the front of \( a_i \) at label \( s_2 \). (See Fig. 12).

Continue in this manner indefinitely, stopping only if

(i) some \( e_{k+1} \) has both ends incident to \( a_i \) (hence both to the front of \( a_i \), since \( f_i \) and \( f_t \) separate the front labels between \( s \) and \( t \) from all the others).

(ii) some \( e_{k+1} \) has its other end at a vertex \( a_j \) with \( j > i \) (so \( a_j \) is not \( a_i \) and not in γ).

This series of edges \( e_1, e_2, e_3, \ldots \), when viewed in \( \Gamma_Q \), is a path beginning at \( b_i \). The tails are always labelled \( i \), so (a) and (b) of 6.5 are always satisfied. (c) is satisfied by construction. Finally, conditions (i) and (ii) imply that \( e_{k+1} \) is larger (in \( \Gamma_Q \)) than \( e_k \). Thus the longest initial segment of the path \( e_1, e_2, e_3, \ldots \) in \( \Gamma_Q \) that satisfies (d) of 6.5 is a special path.

The construction of the special path originating at \( b_i \) is similar.

All vertices of a special path have \( \delta \) the same sign (6.5(a), (c) and 5.5). Since level edges are synchronous, and \( e \) is lobal, \( \delta_i \neq \delta_t \) (via 5.5). Hence the paths must be disjoint.

\[ \]

§7. MULTIFLOWS

**Definition 7.1.** A semi-oriented path is an imbedded path with a given orientation which is never inconsistent with any of the orientations of its edges. For a a semi-oriented path, let \( \tau(\alpha) \) denote the initial vertex and \( \eta(\alpha) \) denote the terminal vertex.

7.2 Let \( e \) be a lobal (or special) edge, with label (of its head) \( i \) at a vertex \( b_i \). Define the large side of \( e \) to be the side on which the adjacent label \( i + 1 \) of \( b_i \) lies. If \( i = 0 \), the large side will be that on which the front of \( b_i \) lies. Note that if an edge \( f \) has tail labelled \( r \) and its head on the front of \( b_i \) and on the large side of an edge \( e \) with head also on the front of \( b_i \), then \( f \) is larger than \( e \) (and so is a terminator if \( e \) is special).

**Definition 7.3.** A multizflow in \( \Gamma_Q \) is a good circuit which is the union of

(a) a collection \( \{\sigma_1, \ldots, \sigma_k\} \) of level edges and/or vertices, called the apexes

(b) 2k semi-oriented paths \( \alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k \) called the currents such that for all \( 1 \leq \lambda \leq k \)

(i) \( \tau(\alpha_\lambda) \) and \( \tau(\beta_\lambda) \) are vertices incident to (or the vertex) \( \sigma_\lambda \).

(ii) \( \eta(\alpha_\lambda) = \eta(\beta_{\lambda+1}) \) and \( \eta(\alpha_\lambda) = \eta(\beta_1) \); these vertices are called the bases of the multizflow.

By convention, an imbedded semi-cycle is regarded as a multizflow with \( k = 0 \).
We will be examining restricted families of multiflows satisfying certain conditions on the apexes, currents, and bases. A semi-cycle will be regarded as being in the family (with \( k = 0 \)) if it satisfies the conditions on currents.

The argument proceeds as follows: First we define a type of multiflow and show that examples exist (assuming, as we have since 3.1, that \( m, n \geq 1 \)). Then we discover properties about the interior of an innermost such multiflow \( \gamma \). The quest for an innermost multiflow requires dividing multiflows by semi-oriented paths and showing that one of the two resulting multiflows still satisfies the apex, current, and base conditions.

A new class of multiflows is then defined, of which \( \gamma \) is an example and for which the apex, current and base restrictions are liberalized. The process is repeated on this new class.

After three such stages we arrive at a multiflow whose properties are contradictory. The contradiction (to \( m, n \geq 1 \)) completes the proof.

**Definition 7.4.** A type \( A \) multiflow \( \gamma \) is one in which

(a) All apexes \( \sigma_\lambda \) are lobal edges.

(b) If there is no larger edge than \( \sigma_\lambda \) incident to either end of \( \sigma_\lambda \) then each end is the origin of an extremal path \( \mu \subset \gamma \).

(c) Any edge is either
   (i) lobal
   (ii) oriented with tail labelled \( r \) or
   (iii) in an extremal path \( \mu \subset \gamma \) which originates at a lobal edge.

(d) The interior of \( \gamma \) lies on the large side of each lobal edge and each extremal edge in \( \gamma \).

(e) On each vertex of a lobal edge or an extremal edge in \( \gamma \), the interior of \( \gamma \) is incident only to the front side and the interior labels are less than \( r \).

**Lemma 7.5.** For a type \( A \) multiflow \( \gamma \):

(i) At least one edge of \( \gamma \) at each base has tail labelled \( r \).

(ii) If \( f \) is an edge with tail labelled \( r \) which lies on or in the interior of \( \gamma \), then \( f \) is incident to an extremal path \( \mu \subset \gamma \) only at its terminus.

(iii) If the initial edge of an arbitrary extremal path \( \rho \) lies on or in the interior of \( \gamma \), so does each edge of \( \rho \).

**Proof.** (i) Follows from 7.4(c) and (d).

(ii) If the head of \( f \) is on the extremal path this follows from 7.4(d) and (e) and 7.2. If the tail of \( f \) (labelled \( r \)) is on the extremal path this follows from 6.5(b) and the fact that \( \mu \) lies entirely in \( \gamma \).

(iii) Suppose \( f \) were the first edge of \( \rho \) not to lie on or in the interior of \( \gamma \). Then the tail of \( f \) lies on the front side (6.5(c)) of a vertex \( b_\tau \) of \( \gamma \) at a label \( \tau \). Since the head of the predecessor \( f' \)
of \( f \) in \( \rho \) is also incident to the front of \( \rho \) (6.5(c)) at a label \( \eta < \tau < r \) (6.5(b)) some edge \( e \) of \( \gamma \) must be incident to a label between \( \eta \) and \( \tau \). The edge \( e \) cannot be a lobal edge or the head of an extremal edge by 7.4(d). It cannot be the tail of an extremal edge, since \( f \) is also extremal with tail at \( b_z \). It cannot have tail labelled \( r \) at \( b_z \) since \( \tau < r \). It cannot have tail labelled \( r \) at another vertex by 7.2. This contradicts 7.4(c) for the edge \( e \).

**Proposition 7.6.** \( \Gamma_e \) contains a type A multiflow. An innermost type A multiflow \( \gamma \) contains in its interior

1. no extremal cycles, nor circuit in which every vertex has an interior or edge label \( r \)
2. no interior labels \( r \)
3. no lobal edges
4. no sources.

**Proof.** Via 6.2(c) there is an imbedded good cycle in \( \Gamma_0 \) with all tails labelled \( r \), hence a type A multiflow (with \( k = 0 \)).

(a) A circuit in which every vertex has an interior or edge label \( r \) has in its interior a cycle with every tail labelled \( r \) (via 6.2(c)), a type A multiflow. If \( \gamma' \) is an extremal cycle then the front side of each edge is either always on the inside or always on the outside of \( \gamma' \), by 6.5(a), (b), (c). In the former case \( \gamma' \) is a type A multiflow (with \( k = 0 \)) and in the latter case every edge has an interior label \( r \).

(b) Let \( \gamma \) be an innermost type A multiflow. If there is an interior label \( r \), then by 6.2(c) and 7.6(a) there is an oriented imbedded path \( \alpha \) in the interior of \( \gamma \) with both ends on \( \gamma \) and each tail of each edge of \( \alpha \) labelled \( r \). Then \( \alpha \) divides \( \gamma \) into two multiflows, one of which \( \gamma' \) still satisfies 7.4(a) because it contains no new apexes. \( \tau(\alpha) \) is not at an apex of \( \gamma \) by 7.4(e), so 7.5(ii) shows \( \gamma' \) still satisfies 7.4(b). Finally, 7.4(c) is true for \( \gamma' \) by construction and 7.4(d) and (e) by default.

(c) Suppose \( f \) is a lobal edge in the interior of \( \gamma \).

**Case i.** The vertices \( b_z \) and \( b_i \) at the ends of \( f \) lie in the interior of \( \gamma \) and there is a larger edge than \( f \) incident to \( b_z \) or \( b_i \).

In this case construct oriented paths \( \alpha' \) and \( \beta' \) with all tails labelled \( r \), beginning at the labels \( r \) on the front side of \( b_z \) and \( b_i \). By 7.6(a) the paths are imbedded, so eventually reach \( \gamma \). Although they may not be disjoint, they can be made never to cross (indeed never to share an edge, though this fact is not used). Each complementary region of \( \alpha' \cup \beta' \) in the interior of \( \gamma \) is bounded by a multiflow; let \( \gamma' \) be that multiflow which contains the large side of \( f \) in its interior. Then the apex \( f \) of \( \gamma' \) satisfies 7.4(a), (d) and (e). The proof in this case now follows as in 7.6(b).

**Case ii.** The vertices \( b_z \) and \( b_i \) lie in the interior of \( \gamma \) and there is no larger edge than \( f \) incident to \( b_z \) or \( b_i \).

They by 6.9 and 6.8(c) there are disjoint extremal paths \( \mu \) and \( v \) in the interior of \( \gamma \) originating at \( b_z \) and \( b_i \). (See Fig. 13.) By 7.5(iii) they lie entirely in \( \gamma' \) or its interior. If the terminus \( b_z \) of \( \mu \) lies in the interior of \( \gamma' \) extend \( \mu \) to a path \( \mu' \), by always exiting at a label \( r \), beginning at the front label \( r \) of \( b_z \). If this extension flows into a vertex \( b_z \) of \( \mu \) (resp. \( v \)) on the large side of the edge of \( \mu(v) \) that flows into \( b_z \), then it terminates \( \mu(v) \). Thus \( \mu' \) is imbedded, for the interior of any circuit would be either a type A multiflow or would contradict 7.6(a), and \( \mu' \) doesn't cross \( \mu \). Similarly extend \( v \) to an imbedded path \( v' \) which doesn't cross \( \mu \). By the appropriate choice of labels \( r \) used in their construction we may assure then that \( \mu' \) and \( v' \)
Fig. 13.

do not cross anywhere. Each complementary region of $\mu' \cup \nu'$ in the interior of $\gamma$ is bounded by a multiflow. As in case i, let $\gamma'$ be the boundary of the complementary region that contains the large side of $f$. The proof in this case now follows as in 7.6(b).

\textit{Case iii.} $b_s$ lies in $\gamma$, but $b_t$ does not.

Proceed as in case ii if there is no larger edge than $f$ incident to $b_s$ or $b_t$. If there is a larger edge, proceed as in case (i) to construct an oriented path $\beta'$ beginning at the label $r$ on the front of $b_s$, and ending on $\gamma$. The union of $f$ and $\beta'$ divides $\gamma$ into two multiflows; let $\gamma'$ be that which contains the large side of $f$.

Let $e_s$ be the edge of $\gamma'$ incident to $b_s$. By 7.4(d) $e_s$ cannot be lobal nor, if it is extremal, can its head be at $b_s$. Thus $e_s$ is either extremal with tail at $b_s$ or has tail labelled $r$ at $b_s$, or has tail labelled $r$, but head at $b_s$. In the first two cases $f$ is an apex of $\gamma'$ and the proof proceeds as in case $i$. In the last case, when $e_s$ has tail labelled $r$, but head at $b_s$, then $\gamma'$ is a type A multiflow in which $f$ is part of a current satisfying 7.4(c). This current still satisfies 7.4(d) and (e) because there is no interior label $r$ at $b_s$ (7.6(b)).

\textit{Case iv.} $b_s$ and $b_t$ both belong to $\gamma$.

If there is no larger edge than $f$ incident to $b_s$ or $b_t$, proceed as in case ii. Suppose there is a larger edge than $f$ incident to $b_s$ or $b_t$. The edge divides $\gamma$ into two multiflows; let $\gamma'$ be that whose interior contains the large side of $f$. If either $e_s$ or $e_t$ has tail labelled $r$ and head at $b_s$ (or $b_t$) then $\gamma'$ is a type A multiflow in which $f$ is part of a current satisfying 7.4(c). The only other possibility, as in case iii, is that each $e_s$ and $e_t$ either is extremal with tail at $f$ or has tail labelled $r$ at $f$. Then $\gamma'$ is a type A multiflow with $f$ a new apex.

(d) Let $b_s$ be a source in the interior of $\gamma$. Then the edge labelled $0$ must be level, hence lobal. This contradicts (c).

§8. TYPE B MULTIFLOWS

\textbf{Definition 8.1.} A type B multiflow is a multiflow $\gamma$, contained in an innermost type A multiflow, for which:

(a) All apexes are either

(i) lobal edges
A PROJECTIVE PLANE IN \( \mathbb{R}^4 \)

(ii) vertices to which an interior edge is incident, either coherent and level or with head at the apex

(iii) the initial vertex for an extremal path \( \mu \subset \gamma \).

(b) Let \( f \) be a lobal edge which is an apex. If there is no larger edge incident to either end of \( f \), then each end is the origin of an extremal path \( \mu \subset \gamma \). If there is a larger edge incident to an end of \( f \), at least one such edge lies in the interior of \( \gamma \).

(c) Each edge is either lobal or oriented, with labels \( < r \).

(d) The interior of \( \gamma \) lies on the large side of each lobal edge and each edge in an extremal path \( \mu \).

(e) At each vertex of an apex, a lobal edge, or an extremal path \( \mu \), the interior of \( \gamma \) is incident only to the front side, and the interior labels are less than \( r \).

(f) A terminator for each extremal path \( \mu \) is in \( \gamma \) or in its interior.

PROPAGATION 8.2. An innermost type A multiflow \( \gamma_A \) is type B. An innermost type B multiflow has no

(a) interior label \( \geq r \)

(b) vertex in its interior

(c) interior oriented terminator of an extremal path \( \mu \) in \( \gamma \)

(d) interior oriented edge with head at an apex.

Proof. \( \gamma_A \) automatically satisfies (a)–(e). To verify (f), let \( b_n \) be the terminator of an extremal path in \( \gamma_A \). The edge \( e \) of \( \gamma \) incident to \( b_n \) but not in \( \mu \) cannot be extremal or lobal, by 7.4(d) and so has tail labelled \( r \). If the tail of \( e \) is on \( b_n \), then a terminator, which has head at \( b_n \) with label \( < r \), must be in the interior of \( \gamma \). If the head of \( e \) is on \( b_n \), then \( e \) itself is a terminator.

(a) First note that it suffices to show that there are no interior labels \( r \). For it then follows from 8.1(c) that at any vertex \( b_n \), on which there is an interior label greater than \( r \), the edges of \( \gamma \) incident to \( b_n \) both would have label \( r \) there. Thus \( b_n \) would be an apex, contradicting 8.1(e).

If there is an interior label \( r \) at a vertex \( b_n \) of \( \gamma \), then construct an imbedded path \( \alpha \) in the interior of \( \gamma \) such that \( \eta(\alpha) = b_n \), \( \eta(\alpha) \) is a vertex \( b_n \) of \( \gamma \), and each edge of \( \alpha \) is oriented with tail labelled \( r \). Denote by \( e \) the first edge of \( \alpha \) and by \( f \) the last edge. (Perhaps \( e = f \).)

Then \( \alpha \) divides \( \gamma \) into two multiflows, both of which continue to satisfy 8.1(f) at \( b_n \) when \( b_n \) is in an extremal path, since \( f \) would be a terminator.

If \( b_n \) is an extremal path \( \mu \) of \( \gamma \), but is not its terminus, then let \( \gamma' \) be the multiflow containing the terminal segment of \( \mu \). The vertex \( b_n \) is the only apex of \( \gamma' \) which is not an apex for \( \gamma \), and it satisfies 8.1(a)(iii). By construction \( \gamma' \) still satisfies 8.1(b) and (c) and satisfies 8.1(d) and (e) because \( \gamma \) did. (See Fig. 14.)

If \( b_n \) is the terminus of an extremal path \( \mu \) of \( \gamma \), choose \( \gamma' \) so that a terminator for \( \mu \) is on \( \gamma' \) or in its interior (8.1(f)). If \( \gamma' \) contains \( \mu \) then it has no new apex and continues to satisfy 8.1(f) by construction, and 8.1(a)–(e) because \( \gamma \) did. If \( \gamma' \) does not contain \( \mu \) then \( b_n \) is a new apex of type 8.1(a)(ii) still satisfying 8.1(e) since \( b_n \) was on an extremal path. 8.1(b)–(d) and 8.1(f) are true for \( \gamma' \) since they were for \( \gamma \).

If \( b_n \) is not on an extremal path, nor at an apex, choose \( \gamma' \) so that it contains the initial segment of the current on which it lies (if it is at a base, choose either current). Then \( \gamma' \) contains no new apex, and satisfies 8.1 because \( \gamma \) did.

If \( b_n \) is in an apex, but not an extremal path, it is in an apex of type 8.1(a)(i) or (ii). Furthermore, if it is in an apex of type 8.1(a)(i), then it is a vertex of a lobal edge to which a larger edge is incident. Choose \( \gamma' \) so it continues to contain this larger edge in its interior. Then \( \gamma' \) has an apex at \( b_n \) either of type 8.1(a)(i) satisfying 8.1(b) or of type 8.1(a)(ii).
If $b_s$ is an apex of type 8.1(a)(ii) then choose $y'$ so it continues to contain the interior edge, incident to $b_s$, which is either coherent level or has head at $b_s$ (and is thus not $e$ itself). Then $b_s$ is still an apex in $y'$ of type 8.1(a)(ii).

(b) There are no sources in the interior, hence no cycles, by 7.6(d). Hence if there is an interior vertex there is an oriented edge $e$ with its tail at a vertex $b_s$ in $y$ and its head at an interior vertex $b_t$. Beginning with a label $r$ on $b_s$, extend $e$ to an imbedded oriented path $\alpha$ in the interior of $y$ such that $\tau(\alpha) = b_s$, $\eta(\alpha)$ is a vertex $b_t$ in $y$, and every edge but $e$ has tail labelled $r$. The proof is now a word for word repeat of (a). (Only the fact that the last edge $f$ of $\alpha$ has tail labelled $r$ was used.)

(c) Suppose $f$ is an oriented terminator of an extremal path $\mu$ in $y$ with head at $b_s$ in $\mu$ and tail at $b_s$. By 8.2(b) we can take $b_s$ to be in $y$, so $f$ divides $\gamma$ into two multiflows. Both of them continue to satisfy 8.1(f) at $b_s$, since $f$ is a terminator of $\mu$. The proof is now a word for word repeat of 8.2(a) (That the last edge $f$ of the path $\alpha$ has tail labelled $r$ was used there only to show that 8.1(f) was satisfied at $b_s$).

(d) Suppose $f$ is an oriented edge with its head at a vertex $b_s$ in an apex and its tail at $b_s$. By 8.2(a) we can take $b_s$ to be in $y$, so $f$ divides $\gamma$ into two multiflows, neither of which any longer has an apex containing $b_s$. The proof is now a word for word repeat of 8.2(a) (Since $b_s$ is an apex of $y$ it is not a terminus).

§9. TYPE C MULTIFLOWS

Definition 9.1. A type C multiflow $\gamma$ is a multiflow, contained in an innermost type $B$ multiflow, in which:

(a) All apexes are either
   (i) vertices to which an interior coherent level edge is incident.
   (ii) the initial vertex for an extremal path $\mu \subset \gamma$.

(b) Each current is a semi-oriented path in which any non-lobal level edge has some predecessor which is oriented.

(c) The interior of $\gamma$ lies on the large side of each lobal edge and each edge of an extremal path $\mu$. 
(d) On each vertex of a lobal edge, apex, or extremal path in $\gamma$, the interior of $\gamma$ is incident only to the front side.
(e) The terminus of each extremal path is either a base of $\gamma$ with terminator on $\gamma$, or there is a coherent level terminator in the interior of $\gamma$.

**Lemma 9.2.** An innermost type B multiflow $\gamma$ is type C.

**Proof.** Suppose $f$ is an apex of type 8.1(a)(i). If there is a larger edge than $f$ incident to one of its ends, $b_\gamma$, then it is coherent level by 8.2(d) so $b_\gamma$ can be viewed as an apex of type 9.1(a)(i) with $f$ the first (lobal) edge of a current. If there is no larger edge than $f$ incident to either end of $f$, then by 8.1(b), $b_\gamma$ is similarly an apex of type 9.1(a)(ii).

Suppose a vertex $b_\gamma$ of $\gamma$ is an apex of type 8.1(a)(ii). Then by 8.2(d) $b_\gamma$ is an apex of type 9.1(a)(i).

A vertex of type 8.1(a)(iii) is already of type 9.1(a)(ii).

Thus 9.1(a) is true for $\gamma$.

9.1(b) is true by default (8.1(c)).

9.1(c) is 8.1(d).

9.1(d) is 8.1(c).

9.1(e) follows from 8.1(f) and 8.2(c), since an oriented terminator must have its head at the terminus.

9.1(f) is 8.2(e).

**Proposition 9.3.** For an innermost type C multiflow $\gamma$:

(a) All apexes are of type 9.1(a)(ii).

(b) The terminus of each extremal path in $\gamma$ is a base at which the other incident edge of $\gamma$ is a terminator.

(c) There is at least one apex.

**Proof.** (a) Suppose $b_\gamma$ is a type 9.1(a)(i) apex with $f$ an interior coherent level edge incident to $b_\gamma$. By 9.1(d) $f$ is incident to the front of $b_\gamma$. By 8.2(b) the other end of $f$ is a vertex $b_\gamma$ of $\gamma$. Furthermore, since $f$ is not lobal (7.6(c)), $f$ cannot be incident to the front of $b_\gamma$, so $b_\gamma$ is not an apex and is not on a lobal edge or extremal path (9.1(d)). (See Fig. 15).

The edge $f$ divides $\gamma$ into two multiflows; choose $\gamma'$ to be that containing the initial segment on which $b_\gamma$ lies, and let $e$ be the edge of $\gamma'$ incident to $b_\gamma$. Since $e$ is not lobal, either it

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Fig. 15.
or a predecessor in the current is oriented (9.1(b)). In the multiflow \( \gamma' \) we then regard \( f \) as lying on the same current as \( e \), so \( b_1 \) is no longer an apex. Then \( \gamma' \) continues to satisfy 9.1(a), (c), and (d) since \( \gamma \) did. It satisfies 9.1(b) by construction and 9.1(e) since neither end of \( \gamma \) is a terminus for an extremal path.

(b) Let \( b_\gamma \) be the terminus of an extremal path \( \mu \) in \( \gamma \). If \( b_\gamma \) is not a base, or if a terminator does not lie in \( \gamma \), then by 9.1(e) there is a coherent level terminator \( f \) in the interior of \( \gamma \). As in (a), the other end of \( f \) is a vertex \( b_\gamma \) in \( \gamma \) which is not an apex, not a lobal edge, and not on an extremal path of \( \gamma \).

The edge \( f \) divides \( \gamma \) into two multiflows; choose \( \gamma' \) to be that containing the initial segment of the current on which \( b_\mu \) lies. Then 9.1(a)–(d) are true for \( \gamma' \) just as in 9.3(a). Finally, 9.1.3 is true for \( \gamma' \) because, if \( \mu \) lies in \( \gamma' \) then \( b_\gamma \) is a base and \( f \) a terminator.

(c) By 9.1(c), adjacent edges of \( \gamma \) cannot be lobal, so by 9.1(b) there is an oriented edge in \( \gamma \). If \( \gamma \) has no apex, it is a semi-cycle. This contradicts 5.7 (via 7.6(c), 9.1(d), 8.2(a), 8.2(b)).

9.4. From 9.3(a) we see that each apex of an innermost type C multiflow \( \gamma \) is the initial vertex of an extremal path, and from 9.3(b) that the extremal path is the entire current. From 9.1(c) we know that two extremal paths cannot meet at a base. We conclude that each base is the terminus of exactly one extremal path in \( \gamma \). Hence we can assume that in the multiflow \( \gamma \) the currents \( \mu_1 \) are all extremal paths. In view of this, we will henceforth denote them \( \mu_\gamma \).

**Lemma 9.5.** For \( f \) an interior edge of \( \gamma \) an innermost type C multiflow:

(a) If \( f \) is oriented, the head lies in an extremal path \( \mu_\beta \).

(b) If \( f \) is level, at least one end lies on a vertex in an extremal path \( \mu_\beta \) other than the base.

Proof. (a) Suppose \( f \) is an oriented edge with its head at a vertex \( b_\gamma \) in a current \( \beta_\gamma \), where \( b_\gamma \) is not an apex or base. The tail of \( f \) is a vertex \( b_\mu \) of \( \gamma \) (8.2(b)), so \( f \) divides \( \gamma \) into multiflows. Let \( \gamma' \) be the one which contains the next edge of \( \mu_\gamma \).

If \( b_\gamma \) lies on a vertex of some \( \beta_\mu \), then it follows from the definition of a multiflow that the terminal segment of \( \beta_\gamma \) and the initial segment of \( \beta_\mu \) lie on the same side of \( f \), and hence are both in \( \gamma' \). (If \( \lambda = \nu \) then \( \gamma' \) is \( f \) together with the segment of \( \beta_\mu \) which lies between \( b_\gamma \) and \( b_\mu \).) It is easy to see that \( \gamma' \) is still a type C multiflow in which the initial segment of \( \beta_\mu \) and the terminal segment of \( \beta_\gamma \) have been joined into a single current by \( f \).

If \( b_\gamma \) lies on a vertex of an extremal path \( \mu_\beta \) then \( f \) follows from the definition of a multiflow that the terminal segments of \( \beta_\gamma \) and \( \mu_\beta \) both lie on the same side of \( f \), hence in \( \gamma' \). It is easy to see that \( \gamma' \) is still a type C multiflow with new apex at \( b_\gamma \).

(b) \( \{ \mu_\gamma \} \) and \( \{ \beta_\mu \} \) intersect only on the apexes and bases. Hence \( b_\gamma \) is a vertex in \( \{ \mu_\gamma \} \) other than a base if and only if it is not a vertex of \( \{ \beta_\mu \} \) except perhaps an apex. Suppose on the contrary, then, that a level edge \( f \) in the interior of \( \gamma \) has ends at \( b_\gamma \) in \( \beta_\mu \) and \( b_\gamma \) in \( \mu_\beta \) but neither vertex is an apex.

The immediate predecessors \( e_\gamma \) for \( b_\gamma \) in \( \beta_\mu \) and \( e_\mu \) for \( b_\gamma \) in \( \mu_\beta \) lie on opposite sides of \( f \), by 7.3. Since level edges are synchronous, 9.1(c) and 9.1(d) imply that \( e_\gamma \) and \( e_\mu \) can't both be lobal. Similarly \( b_\gamma \) and \( b_\gamma \) can't both be bases (9.1(c)), and if \( b_\gamma \) is a base then \( e_\gamma \) is not lobal (9.1(c)).

So take \( e_\gamma \) to be a non-lobal edge, with \( h_\gamma \) a base if either \( h_\gamma \) or \( b_\gamma \) is. Then \( b_\gamma \) is not a base and some predecessor of \( e_\gamma \) is oriented (9.1(b)). Let \( \gamma' \) be the multiflow lying on the same side of \( f \) as \( e_\gamma \). Then \( \gamma' \) is a type C multiflow in which the initial segment of \( \beta_\mu \) and the terminal segment of \( \beta_\gamma \) have been joined into a single current by \( f \).

Let \( q \) be the largest label of any tail appearing in any of the \( \{ \mu_1, \ldots, \mu_k \} \) in \( \gamma \), say in \( \mu_k \).

**Lemma 9.6.** (a) The first edge in \( \beta_1 \) has both ends labelled less than \( q \).
(b) The last edge in $\beta_1$ has tail (or both ends, if level) greater than $q$.

(c) Any interior label of $\gamma$ on a vertex $b_*$ in $\{\mu_*\}$ other than a base is less than $q$.

Proof. (a) The first edge in $\mu_1$ has tail with special label $\leq q$ at the apex $\sigma_1$ by 6.5(a). Then the edge is asynchronous, so the result follows from 9.1(c) and (d).

(b) By 6.5(b) the last edge in $\mu_q$ has tail labelled $q$. Since the last edge in $\beta_1$ can't be lobal (9.1(c)) the result follows from 9.3(b).

(c) This follows from 6.5(b) and (c).

Let $l_1, \ldots, l_j$ be the edge and interior labels of $\beta_1$ read in order along $\beta_1$ beginning with the label $l_1 < q$ of 9.6(a) and ending with the label $l_p > q$ of 9.6(b). (See Fig. 16).

**Lemma 9.7.** Let $l_j$ be the last label for which $l_j < q$. Then $l_j + 1 = q$ and $l_j + 1$ is an interior label of $\gamma$.

**Proof.** Since any oriented edge of $\beta_1$ points from an $l_i$ to $l_{i+1}$, and $l_{i+1} \geq q > l_i$, $l_j$ and $l_{j+1}$ cannot be the ends of an edge lying in $\beta_1$. Therefore $l_{j+1}$ and $l_j$ are adjacent on a common vertex. Since $l_{j+1} \geq q$, it follows that $l_j + 1 = q$ and so $j + 1 < p$. Similarly, if $l_{j+1}$ and $l_{j+2}$ are the ends of an edge in $\beta_1$, it must be a level edge. But this, too, is impossible since a level edge is synchronous, forcing $l_{j+3} = l_j < q$ and contradicting the choice of $j$. Hence $l_{j+1}$ is an interior label.

9.8 Proof of 1.3. Following 3.1 we have assumed $m, n \geq 1$. Here is a contradiction: Let $f$ be the edge in the interior of an innermost type $C$ multiflow $\gamma$ with end the label $l_{j+1}$ of 9.7. Then the label is $q$ and lies on a vertex $b_*$ not in any $\mu_*$. By 9.5(a), $f$ is not oriented with head at $b_*$. By 9.5(b) and 9.6(c) $f$ is not level. Hence $f$ is oriented with tail at $b_*$. Then $f$ is asynchronous (6.5(a)), so $l_{j+2} < q$ (9.1(c)). This contradicts the choice of $j$.

**References**

