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PRODUCING REDUCIBLE 3-MANIFOLDS BY SURGERY ON A KNOT

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IT HAS long been conjectured that surgery on a knot in S^3 yields a reducible 3-manifold if and only if the knot is cabled, with the cabling annulus part of the reducing sphere (cf. [7, 8, 9, 10, 11]). One may regard the Poenaru conjecture (solved in [5]) as a special case of the above. More generally, one can ask when surgery on a knot in an arbitary 3-manifold M produces a reducible 3-manifold M'. But this problem is too complex, since, dually, it asks which knots in which manifolds arise from surgery on reducible 3-manifolds. In this paper we are able to show, approximately, that if M itself either contains a summand not a rational homology sphere or is ∂-reducible, and M' is reducible, then k must have been cabled and the surgery is via the slope of the cabling annulus. Thus the result stops short of proving the conjecture for $M = S^3$, but (see below) does suffice to prove the conjecture for satellite knots.

The results here are broader than this; for a context recall the main result of [3]:

GABAI'S THEOREM. *Let k be a knot in $M = D^2 \times S^1$ with nonzero wrapping number. If M' is a manifold obtained by non-trivial surgery on k then one of the following must hold:*

(1) *$M = D^2 \times S^1 = M'$ and both k and k' are* 0 *or* 1*-bridge braids.*
(2) *$M' = W_1 \# W_2$, where W_2 is a closed 3-manifold and $H_1(W_2)$ is finite and non-trivial.*
(3) *M' is irreducible and $\partial M'$ is incompressible.*

It is this theorem we generalize to many other manifolds M. We also examine case (2) and show that it only arises (i.e. M' is only reducible) if k is cabled with cabling annulus having the slope of the surgery. (For a detailed look at case (1), see [6] or [1].) Specifically we have:

THEOREM. *Let M be a compact orientable 3-manifold. Suppose k is a knot in M with $M - k$ irreducible and ∂-irreducible. Let M' be a manifold obtained by Dehn surgery on M, with $k' \subset M'$ the core of the filling torus. If ∂M compresses in M or M contains a sphere not bounding a rational homology ball then either*

(a) *$M' = D^2 \times S^1 = M$ and both k and k' are* 0 *or* 1*-bridge braids*
(b) *$M' = D^2 \times S^1$, $M = D^2 \times S^1 \# L$ for some Lens space L, k is the knot sum of the core of L and a* 0*-bridge braid in $D^2 \times S^1$, and k' is the cable on a* 0*-bridge braid.*
(c) *k is cabled and the slope of the surgery is that of the cabling annulus or*

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(d) *M' is irreducible. No torus component of ∂M compresses in both M and M'. Any pair of simple closed curves $c, c' \subset \partial M$ which compress in M and M' respectively must intersect. In fact, if M contains a sphere not bounding a rational homology ball, M' is ∂-irreducible.*

Remarks. Gordon–Litherland [8] give examples of two reducible 3-manifolds M and M', each obtained from the other by surgery on a non-cabled knot. But in each the unique reducing sphere bounds a rational homology ball. This shows we need the assumption that some reducing sphere in M bounds no rational homology ball.

In case (d) we cannot hope to conclude that M' is also ∂-irreducible. Indeed, take for M a solid handlebody and for k a knot parallel, via an annulus A, to a curve c in ∂M for which $\partial M - c$ is incompressible. Let M' be obtained by surgery on k with slope that of $\partial A \cap \dot{\eta}(k)$. Then clearly c compresses in M' yet M and M' satisfy none of (a), (b), (c) above.

A sample application of the theorem:

4.5 Corollary. *If surgery on a satellite knot k in S^3 yields a reducible 3-manifold, then k is cabled.*

We assume familiarity with the terminology of [13].

Here is an outline of the paper. §1 and 2 are fairly technical. The goal is to show that if a β-taut sutured manifold has a particular kind of disk or sphere in it, then one can construct, as in [13], a β-taut sutured manifold hierarchy for which every term contains such a disk or sphere. On first reading, Definition 1.1 and Theorem 2.5 suffice for understanding the rest of the paper.

In §3 we formalize an object first used by Gabai in his proof of the Poenaru conjecture [5]. This "Gabai disk" is used here in two entirely distinct ways: first, as in [5], it guarantees the existence of an x-cycle (terminology from [2]) which in turn provides a new homologous surface intersecting the knot in fewer points. Second, an Euler characteristic argument on the graph inside the Gabai disk provides a cabling annulus for k (see 3.4).

§4 can be regarded as the heart of the argument and 4.3 as the central theorem. Its proof exploits sutured manifold hierarchies of the special type created in §1 and §2. Hierarchies are needed on the manifold both before and after surgery.

§5 prepares to translate the results of §4, which are results about sutured 3-manifolds, into standard 3-manifold notions of ∂-reducing disks and reducing spheres.The central point is that a 3-manifold with a pair of disjoint simple closed curves on its boundary can usually be given a taut sutured manifold structure for which the curves are disjoint from the sutures. This is perhaps of independent interest.

§6 then contains the proof of the main theorem stated above.

§1. ADMISSIBLE DISKS AND SPHERES

1.1 *Definition.* Suppose (M, γ) is a β-taut sutured manifold. A *special torus* is a torus $T \subset M$ in general position with respect to β and having a compressing disk D so that the decompositions $(M, \gamma) \xrightarrow{T} (M', \gamma') \xrightarrow{D} (M'', \gamma'')$ are β-taut. A proper disk $(D, \partial D) \subset (M, R_\pm)$ is *admissible* if it compresses a component of $R_\pm$ other than a special torus. A sphere in M is *admissible* if it does not bound a rational homology ball in M. Note that since M is β-irreducible and $R_\pm$ is β-incompressible, any admissible disk or sphere intersects β. Note also that if a 2-handle in M is attached to an admissible 2-sphere, one of the two resulting 2-spheres is also admissible.

The goal of this section and the next is to show that if (M, γ) has an admissible disk or sphere then it has a β-taut sutured manifold hierarchy with an admissible disk or sphere at each stage. In this section we show how to construct a single decomposition preserving the existence of an admissible disk or sphere. In the next we show how to complete an entire hierarchy.

1.2 LEMMA. *Suppose (M, γ) is a β-taut sutured manifold with $H_2(M, \partial M) \neq 0$, and E is an admissible disk or sphere for M. Then there is a β-taut sutured manifold decomposition $(M, \gamma) \to (M', \gamma')$ along a non-separating surface S so that (M', γ') also admits an admissible disk or sphere.*

Moreover, the decomposition by S may be made to respect a given parameterizing surface in M.

Proof. If $\operatorname{image}(H_2(M) \to H_2(M, \partial M)) \neq 0$, choose a β-taut closed non-separating connected surface S in M. Alter E rel ∂E so that among all such choices $|S \cap E|$ is minimized. Let (M', γ') be the sutured manifold obtained by decomposition along S. Since S is closed and β-taut, (M', γ') is β-taut. If S is disjoint from E, then E remains admissible in M'. If some component of $S \cap E$ were inessential in S, then a disk switch along an innermost such circle in S would alter E, lowering $|S \cap E|$. So S intersects E only in circles essential in S. An innermost disk E' of E cut off by S is a compressing disk for S in M. If S is not a torus, then E' is an admissible disk for M'. If S is a torus, then S compressed along E' is an admissible sphere for M', since S is non-separating.

Henceforth we therefore make:

Assumption A. ∂: $H_2(M, \partial M) \to H_1(\partial M)$ is injective and non-trivial, i.e. M contains non-separating surfaces, but none of these are closed.

Claim 1. There is a non-empty disjoint collection C of closed oriented simple closed curves in ∂M such that

(a) C bounds a non-separating surface in M (i.e. $[C] \neq 0$ is in $\operatorname{image}(\partial: H_2(M, \partial M) \to H_1(\partial M))$

(b) C is disjoint from ∂E

(c) any given component of γ intersects every component of C with the same orientation

(d) no subcollection of C is null-homologous in ∂M.

Proof of Claim 1. If E is non-separating then by Assumption A, E is a disk and ∂E is non-separating in ∂M. Then just take for C a curve parallel to ∂E. So assume E is separating.

At least one component M_0 of $M - E$ has $H_2(M_0, \partial M_0) \neq 0$; let $(T, \partial T) \subset (M_0, \partial M_0 - E)$ be a non-separating surface in M_0. Then ∂T satisfies (a) and (b). If some component of γ intersects ∂T in points of opposite sign, tube together adjacent points of opposite sign in γ, cancelling them all by pairs. This modifies T until ∂T satisfies (c) as well. Finally, remove any collection of components of ∂T which is null-homologous in ∂M. This doesn't change $[\partial T]$ in $H_1(\partial M)$ so (a) is still satisfied, but ensures that ∂T satisfies (d) as well. This proves the claim.

Proof of 1.2 (*resumed*). According to [13, 2.5 and 2.6] there is a connected taut surface $(S, \partial S) \subset (M, \partial M)$ such that

(a) ∂S lies in the oriented train-track given by $C \cup \gamma$

(b) ∂S passes at most once through any arc of $C - \gamma$

(c) S induces a taut-sutured manifold decomposition $(M, \gamma) \to (M', \gamma')$ respecting the given parameterizing surface.

Let $\partial_0 M$ denote the union of all components of ∂M containing elements of C along which some component of ∂S runs. By Claim 1(d) no component of $\partial_0 M$ is a sphere.

Case 1. There is a component γ_0 of γ which bounds a disk in $\partial_0 M$.

With no loss, assume γ_0 bounds a disk D_- lying entirely in R_-. We may assume ∂S is disjoint from D_-, though ∂S may contain some components parallel (with orientation) to γ_0 in R_+. Since no component of $\partial_0 M$ is a sphere, γ_0 does not bound a disk in R_+. Neither does it bound a disk in R'_+. Indeed, if γ_0 bounded a disk D in R'_+, which is the union of S and pieces of R_+ glued together along components of $\partial S \cap R_+$, D would have to contain S. But then $D \cup D_-$, when pushed slightly into M, would be a non-separating closed surface in M, contradicting Assumption A.

Since γ_0 bounds no disk in R'_+, a curve parallel to γ_0 in R'_+ is incompressible in R'_+ but bounds a disk in M parallel to D_-, which is then an admissible disk for M'. Thus the Lemma is satisfied in this case.

So henceforth make

Assumption B. $\partial_0 M$ contains no inessential sutures. In particular, if ∂E lies on $\partial_0 M$, it is essential.

Claim 2. Every component α of ∂S is essential in ∂M.

Proof of claim. A component α is carried by the traintrack determined by $\alpha \cup C$, and so is either parallel to a suture, parallel to a closed component of C above, or, by (c) of claim 1 has non-trivial algebraic intersection with some suture. Since by (B) there are no inessential sutures, and by Claim 1(d) there are no inessential components of C, α must be essential in ∂M, proving claim 2.

Case 2. S is disjoint from E.

Then E persists in M'. If E was an admissible sphere in M then it remains admissible in M'.

∂E does not compress a special toral component T of $R(\gamma')$ because if it did then either $T \subset R(\gamma)$ or $S \subset T$. The latter is impossible since T would then be a non-separating closed surface in M. Therefore if E is not admissible in M' then ∂E bounds a disk F in R'_+, say. R'_+ is the union of S and pieces of R_+, attached along components of $\partial S \cap R_+$. Moreover it cannot lie entirely in R_+, since ∂E is essential in R_+. Consider an innermost circle α of $\partial S \cap R_+$ in F. By Claim 2 α must bound a component of S. Since S is connected, S is a disk and $F - S$ is an annulus between ∂S and ∂E in R_+.

In $\partial M'$, S appears twice, once in F and once as a component S_- of R'_- bounded by a suture γ. If γ does not bound a disk in R'_+, then a push-off of S_- is an admissible disk in M'. If γ does bound a disk F' in R'_+, then, as above, F' consists of an annulus A in R_+ attached to S. In M, the two boundary components of A are both ∂S, and so comprise a torus component T_0 of R_+ in ∂M, for which E and S are both compressing disks. But the compression by S shows T_0 is special, contradicting our assumption that E is admissible.

Case 3. S intersects E.

Since ∂S is in the train-track defined by $C \cup \gamma$, ∂S is disjoint from E. Hence all components of intersection are simple closed curves. If any such curve were inessential in S, we could do a disk-swap along an innermost circle of intersection in S, altering E so that it intersects S in fewer components. Hence we can assume every component of $S \cap E$ is essential in S. (In particular, S is not a disk.)

An innermost circle of $S \cap E$ in E bounds a disk E' in E which compresses S and whose boundary lies entirely in $R'_{\pm}$. We claim E' is an admissible disk for $R'_{\pm}$. For otherwise $\partial E'$ bounds a disk F in $R'_{\pm}$, but not one entirely in S. Let α be an innermost circle of $\partial S \cap R_+$ in F. The disk in F it bounds cannot lie in R_+ by Claim 2, nor in S, since S is connected and not a disk. The contradiction proves the claim. ||

§2. HIERARCHIES

We wish to use 1.2 to construct, for (M, γ) a β-taut sutured manifold admitting an admissible disk or sphere and Q a parameterizing surface for M, a β-taut sutured manifold hierarchy for (M, γ) respecting Q so that every manifold in the hierarchy contains an admissible disk or sphere. In principal this should follow easily from 1.2 but recall from [13] that in a hierarchy, decompositions along product annuli and disks may be required. A decomposition along a product annulus one of whose ends is inessential in $R(\gamma)$ may produce a sutured manifold without admissible disks. This is a technical glitch which forces us to alter the hierarchy slightly when such a product annulus is encountered, replacing the annulus with the disk obtained from it by capping off its inessential end. We will show that the disk together with some arc amalgamations is as effective as the annulus at eliminating index zero disks, which is its role in the hierarchy.

So we modify slightly the definitions from [13] and show that the central properties from [13] still hold.

2.1 *Definition.* A β-taut sutured manifold hierarchy is a finite sequence $(M_0, \gamma_0) \xrightarrow{S_1} (M_1, \gamma_1) \xrightarrow{S_2} \cdots \xrightarrow{S_n} (M_n, \gamma_n)$ of β-taut sutured manifold decompositions for which

(a) each S_i is either a conditioned surface, a product disk, a product annulus both of whose ends are essential in $R(\gamma_{i-1})$, or a disk whose boundary is β-essential in $R(\gamma_{i-1})$

(b) no closed component of any S_i separates

(c) $H_2(M_n, \partial M_n) = 0$, so in fact ∂M_n is a union of spheres.

2.2 *Definition*: Let (M, γ, β) be a suture manifold and β_0 an arc component of β with one end in each of $R_{\pm}(\gamma)$. Let (M', γ', β') be the sutured manifold obtained from (M, γ, β) by setting $M' = M - \eta(\beta_0)$, $\beta' = \beta - \beta_0$, and $\gamma' = \gamma \cup (\text{core of annulus } \dot{\eta}(\beta_0))$. Then say (M', γ', β') is obtained from (M, γ, β) by *converting* the arc component β_0 of β into a suture.

2.3 LEMMA. *Suppose (M', γ', β') is obtained from (M, γ, β) by converting β_0 into a suture. Then (M', γ', β') is β'-taut if and only if (M, γ, β) is β-taut.*

Proof. By excision, $H_2(M', \gamma') \cong H_2(M, \gamma \cup \eta(\beta_0)) \cong H_2(M, \gamma)$. Let $(S, \partial S) \subset (M, \partial M)$ be a β-taut surface homologous rel $\partial S = \gamma$ to $R_{\pm}(\gamma)$. Then S intersects β_0 precisely once, since $R_{\pm}$ does, and $S' = S - \eta(\beta_0) \subset (M', \partial M')$ is homologous to $R_{\pm}'$ rel $\partial S' = \gamma'$. Similarly, if $(S', \partial S') \subset (M', \eta(\gamma'))$ is homologous to $R'_{\pm}$ rel $\partial S' = \gamma'$, then filling in a meridian disk of $\eta(\beta_0)$ produces a surface $(S, \partial S) \subset (M, \partial M)$ which intersects β_0 exactly once. Furthermore $\chi_{\beta}(S) = \chi_{\beta'}(S')$. Thus $R_{\pm}$ is β-norm minimizing if and only if $R'_{\pm}$ is β'-norm minimizing.

Essentially by definition, $R_\pm$ is β-incompressible if and only if $R'_\pm$ is β'-incompressible and M is β-irreducible if and only if M' is β'-irreducible. ||

2.4 Suppose $(M, \gamma) \xrightarrow{A} (M', \gamma')$ is a β-taut sutured manifold decomposition along a non-trivial product annulus, such that one (or both) ends of A are inessential in $R(\gamma)$. Let D be the disk obtained by capping off an inessential end of A in $R(\gamma)$ and pushing slightly into M. Let (M'', γ'') be the sutured manifold obtained by decomposing along D.

LEMMA. *After a series of amalgamations of arcs of β'' in M'', and the conversion of an arc to a suture, we have $(M'', \gamma'', \beta'') \cong (M', \gamma', \beta')$.*

In particular, $(M, \gamma) \xrightarrow{D} (M'', \gamma'')$ is β-taut and, after some arc amalgamations (M'', γ'') has the same index zero disks and reduced complexity as (M', γ').

Proof. Let $E \subset R(\gamma)$ be a disk bounded by an inessential end δ of A. Choose the orientation of A so that δ is oriented oppositely by A and E, so A and E induce the same orientation of $A \cup_\delta E$. Let E' be the disk gotten by pushing E into $\operatorname{int}(M)$, so $D = A \cup_\delta E'$, and recall M'' is the manifold obtained by decomposing along D. Between E and E' in M'' lie a collection of parallel arcs of β'', one for each end of β lying in E. Amalgamate them into a single arc and convert it to a suture. The result is the same sutured manifold as (M', γ', β') (see Fig. 2.1).

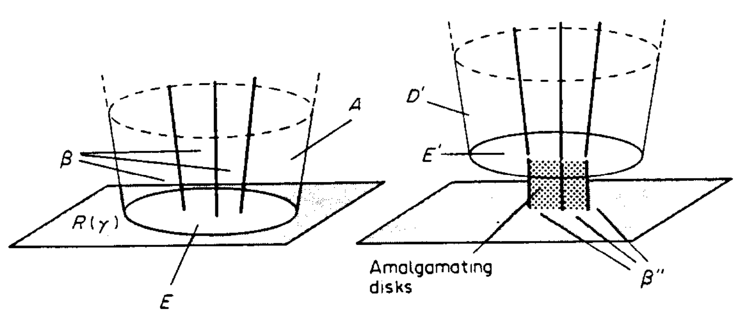

Fig. 2.1.

We know from [13, 4.4] that arc amalgamation does not affect β-tautness and from 2.3 that arc conversion doesn't. Finally, the definition of index zero disks in [13, 4.6] does not distinguish between meridians of arc components of β and sutures, so the arc conversion does not introduce or eliminate index zero disks. ||

2.5 THEOREM. *Any β-taut sutured manifold (M, γ) admits a β-taut sutured manifold hierarchy respecting a given parameterizing surface.*

Moreover, if (M, γ) has an admissible disk or sphere then such a hierarchy can be found so that every term in the decomposition has an admissible disk or sphere.

Proof. The proof is a variant of [13, 4.19]. First we will eliminate index zero disks and verify that afterwards the manifold contains an admissible disk or sphere. Then we will decompose along a non-separating surface chosen via 1.2 and show that the result again contains an admissible disk or sphere.

Let E denote an admissible disk or sphere in M.

Eliminating index zero non-self-amalgamating disks. First eliminate as many index zero disks in (M, γ) as possible without using self-amalgamating disks. This process may require decomposition along a product disk S. Such a product disk S intersects E in a 1-manifold with all its ends on one end of S, since $\partial E \subset R(\gamma)$. If $E \cap S$ consists of simple closed curves, then an innermost disk argument provides an admissible sphere or disk E' disjoint from S with $\partial E' = \partial E$. (Decomposition along S cannot produce a new component of ∂M without sutures, so E' still cannot compress a special toral component of $R(\gamma')$.) If $E \cap S$ contains intervals, then E is an admissible disk and an outermost arc argument in S provides a compressing disk for $R(\gamma')$ whose boundary lies in a component of $\partial M'$ containing a copy of S, hence sutures. Thus again E' does not compress a special toral component of $R(\gamma')$ and so is admissible.

Eliminating self-amalgamating disks. To eliminate self-amalgamating disks we need to decompose along product annuli which are non-trivial (i.e. none just cuts off a copy of $D^2 \times I$ disjoint from β).

Suppose first that A is a product annulus with both ends essential in $R(\gamma)$. If A is incompressible in M then an innermost disk, outermost arc argument as above shows there is an admissible disk or sphere E' in M disjoint from A. If E' is an admissible disk in M then as in Claim 2, Case 2 in the proof of 1.2 E' remains admissible in M'. (Any toral component of $R(\gamma')$ belongs to $R(\gamma)$ since A will produce a suture in $R(\gamma')$ at each of its ends.) If A is compressible in M, then a compressing disk becomes an admissible disk in M'.

So now assume an end of A is inessential in $R(\gamma)$. Cap it off as in 2.4, decompose along the resulting disk D, and amalgamate arcs as in 2.4. By 2.4 the effect on β-tautness, index zero disks, complexity, and parameterizing surfaces is as if we had decomposed along the original product annulus. Moreover an innermost disk outermost arc argument as above shows we may take E in M to be disjoint from D.

If ∂D is essential in $R(\gamma)$ then as in Claim 2, Case 2 in the proof of 1.2 either D is a β-taut compressing disk for a torus component of $R(\gamma)$, or after the decomposition a push-off D' of D is the required admissible disk for M' not compressing a special toral component of $R(\gamma')$. In the first case, ∂D lies on a special toral component of $R(\gamma)$. Since ∂E does not, decomposition along D leaves E an admissible disk or sphere in M'.

Finally, if ∂D is inessential in $R(\gamma)$ then we had a choice which end of A to cap off in the construction of D from A. In particular, we can guarantee that ∂D and ∂E do not lie on the same component of $R(\gamma)$. Then decomposition along D leaves E admissible in $R(\gamma')$.

Continue the process until all index zero disks are eliminated. Note that in eliminating index zero disks as above, any decomposing product annulus has both ends essential in $R(\gamma)$ and any decomposing disk is either a product disk or has its boundary β-essential in $R(\gamma)$, so the decompositions satisfy 2.1(a). As in the proof of [13, 4.19], the final sutured manifold (M_1, γ_1) has complexity $C(M_1, \gamma_1) = \hat{C}(M_1, \gamma_1) = \hat{C}(M, \gamma)$ and (M_1, γ_1) is still β_1-taut.

Decomposing along a non-separating surface. The argument now proceeds as in [13, 4.19]: if ∂M_1 is not a union of spheres then β-taut decomposition along any surface not boundary parallel will decrease complexity. According to 1.2 there is a β-taut sutured manifold decomposition $(M, \gamma) \xrightarrow{S} (M', \gamma')$ respecting the parameterizing surface so that (M', γ') contains an admissible disk or sphere and S is non-separating. In particular the decomposition decreases sutured manifold complexity. Begin the argument again on M' and continue until $H_2(M_n, \partial M_n) = 0$. ||

Now we show that this modified notion of hierarchy still satisfies the following critical property: If a β-taut sutured manifold hierarchy for (M, γ) terminates in a manifold (M_n, γ_n) which is also taut in the Thurston norm (denoted *ϕ-taut*) then either a component of M is a solid torus with no sutures or every term in the decomposition is also ϕ-taut. We need the following variant of [13, 3.9]:

2.6 LEMMA. *Suppose* $(M_0, \gamma_0) \xrightarrow{S_1} (M_1, \gamma_1) \xrightarrow{S_2} \cdots \xrightarrow{S_n} (M_n, \gamma_n)$ *is a sequence of sutured manifold decompositions in which*

(a) *no component of* M_0 *is a solid torus disjoint from* β *and* γ_0

(b) *each* S_i *is either a conditioned surface, a product disk disjoint from* β*, a product annulus disjoint from* β *with both boundary components* β*-essential in* $R(\gamma_{i-1})$ *or a disk* D *such that*

(i) $\partial D \subset R(\gamma_{i-1})$

(ii) *If* ∂D *is* β*-inessential in* $R(\gamma_{i-1})$ *then* D *is disjoint from* β

(c) *no closed component of any* S_i *separates.*

Then if (M_n, γ_n) *is* β*-taut, so is every decomposition in the series.*

Proof. The proof follows from [13, 3.9] unless some S_i is a disk D as in *b*) for which $(M_{i-1}, \gamma_{i-1}) \xrightarrow{S_i} (M_i, \gamma_i)$ is not β-taut. Let $D = S_i$ be the last for which $(M_{i-1}, \gamma_{i-1}) \xrightarrow{S_i} (M_i, \gamma_i)$ is not β-taut. Then we know each term in the series $(M_i, \gamma_i) \xrightarrow{S_{i+1}} (M_{i+1}, \gamma_{i+1}) \ldots \xrightarrow{S_n} (M_n, \gamma_n)$ is β-taut, and in particular (M_i, γ_i) is β-taut.

If ∂D is β-inessential in $R(\gamma_{i-1})$ then by b(ii), D is disjoint from β. Consider the component V of ∂M_i which is the union of D and the subdisk of $R(\gamma_{i-1})$ disjoint from β which ∂D bounds. This is a sphere disjoint from β, hence bounds a ball component N of M_i disjoint from β. If V contains no sutures it lies entirely in $R(\gamma_i)$ and compresses via N, contradicting β-tautness of M_i. Otherwise V contains exactly ∂D as a suture and N can be viewed as $D \times I$ with its natural sutured structure, i.e. just a collar on D. But then $(M_{i-1}, \gamma_{i-1}) \cong (M_i, \gamma_i)$ contradicting the assumption that one is taut and the other not.

If ∂D is β-essential then from [13, 3.6] some component of M_{i-1} is a solid torus disjoint from β_i and γ_i. Hence by (a) there is a decomposition $(M_{j-1}, \gamma_{j-1}) \xrightarrow{S_j} (M_j, \gamma_j)$ for which the number of components which are solid tori disjoint from β_j and γ_j increases.

Let $(M_{j-1}, \gamma_{j-1}) \xrightarrow{S_j} (M_j, \gamma_j)$ be the last such decomposition. As in the proof of [13, 3.9], S_j can be neither a conditioned surface nor a product annulus, and so must be a disk D with ∂D essential in $R(\gamma_{j-1})$. Then decomposing along D creates a solid torus W disjoint from β and γ, so D must separate M_{j-1} and that component of M_j containing the copy of D having no suture on its boundary must be W. In particular, D is disjoint from β_{j-1}. Let M' be the other component of M_j containing D and consider the restriction of the rest of the decomposition to M'. Since no further decompositions increase the number of solid tori disjoint from β and γ, the argument above shows M' is β_j-taut. But $\partial M'$ has a suture bounding D, disjoint from β_j. Then to be β_j-taut, M' must be a ball disjoint from β containing a single suture, i.e. just the product $D \times I$. As above, $(M_{j-1}, \gamma_{j-1}) \cong (M_j, \gamma_j)$, contradicting the choice of j. ∥

2.7 COROLLARY. *Suppose a β-taut sutured manifold hierarchy for (M, γ) terminates in (M_n, γ_n). If (M_n, γ_n) is ϕ-taut then either a component of M is a solid torus whose boundary lies in $R(\gamma)$ or else (M, γ) and every decomposition in the series is ϕ-taut.*

Proof. By definition of a hierarchy the β-taut hierarchy satisfies 2.6(b) and (c) if we replace β by ϕ. The corollary follows. ||

§3. GABAI DISKS

3.1 *Definitions*: Let M be a compact 3-manifold containing a knot k. Let μ denote a meridian curve in $\dot{\eta}(k)$ and λ denote a curve in $\dot{\eta}(k)$ intersecting μ once. A simple closed curve c in $\dot{\eta}(k)$ for which $c \cdot \mu = s$ and $c \cdot \lambda = r$ is said to have *slope* $r/s \in \mathbf{Q}/\mathbf{Z}$. (Ambiguity in the choice of λ forbids taking r/s in $\mathbf{Q}$ unless there is a preferred longitude, e.g. M a homology sphere). A manifold M' is obtained from M by doing Dehn surgery with *slope c* if $\eta(k)$ in M is replaced with a solid torus $S^1 \times D^2$ in M' so that ∂D^2 is isotopic to c. The core $S^1 \times \{0\}$ of the filling torus $S^1 \times D^2$ is a knot denoted $k' \subset M'$. Note that a meridian circle of $\eta(k')$ in $\dot{\eta}(k)$ intersects a meridian of $\eta(k)$ in s points.

Let $k \subset M$ and $k' \subset M'$ be as above, with slope of the surgery r/s. Let Q be a sphere or properly imbedded disk in M' intersecting k' transversally in $p > 0$ points. Suppose D is any disk in $\operatorname{int}(M)$ intersecting Q and k transversally. Q and k give rise to a "graph" Γ in D with vertices the points $k \cap D$, edges the arcs of $Q \cap D$, and circles the simple closed curves of $Q \cap D$. The valence of each vertex in Γ is $p \cdot s$. A loop or circle in Γ is *trivial* if the subdisk of D it bounds contains no other part of Γ. A component of $D - \Gamma$ is *bounded* if its boundary lies completely in Γ, i.e. its boundary is disjoint from ∂D.

3.2 *Definition.* A disk $D \subset \operatorname{int}(M)$ is a *Gabai disk* with respect to $Q \subset M'$ if

(a) $|k \cap \operatorname{int}(D)| = q > 0$ and all points of intersection have the same orientation and
(b) $|Q \cap \partial D| < p \cdot s$, i.e. Γ has fewer than $p \cdot s$ end points on ∂D.

3.3 LEMMA. *Suppose $M' - \eta(k')$ is irreducible. Let Q' be a sphere (resp. proper disk) in M' such that $k' \cap Q' \neq \phi$ and $Q' - \eta(k')$ is incompressible in $M' - \eta(k')$.*

If there is a Gabai disk for Q' in M, then there is a sphere (resp. proper disk with the same boundary) P in M' such that P is disjoint from Q', P is homologous (rel ∂) to Q' and $|P \cap k'| < |Q' \cap k|$. Moreover the region between P and Q' is a rational homology $S^3 \times I$ (resp. rational homology ball).

Proof. This is essentially [13, 9.3] and is implicit in [2, 2.5.2, 2.6.1 and 2]. ||

3.4 LEMMA. *Suppose Q' is a separating sphere in M' with given complementary components X and Y. If a Gabai disk exists for Q' in M then either*

(a) *the interior of some trivial loop or circle of Γ lies in Y*
(b) *a bounded region of $D - \Gamma$ lies in X or*
(c) *$p = 2$, $s = 1$, k is cabled, $Y \cap M$ is a solid torus and the slope of the surgery is that of the cabling annulus.*

Proof. We can assume there are no trivial loops or circles in Γ, for if an innermost one bounds a region in Y this is conclusion (a); if it lies in X we have (b).

Let e denote the number of edges of Γ with both ends on vertices in the Gabai disk D and $e' \leq ps - 1$ denote the number of edges with exactly one end on a vertex. Then $psq = 2e + e' \leq 2e + ps - 1$ so

(1) $ps(q - 1) \leq 2e - 1$.

Let f denote the number of bounded faces in $D - \Gamma$. Let Λ denote the complex which is the union of all bounded faces and interior edges. Since Λ contains all bounded regions it is simply connected, and

(2) $q - e + f \geq 1$.

If any edge belongs to two faces of Λ then one such face lies in X yielding conclusion (b). Otherwise, since there are no trivial loops, $e \geq 2f$ and from (2) we then get $q - e/2 \geq 1$ or

(3) $4(q - 1) \geq 2e$

Combining (1) and (3) gives $ps(q - 1) \leq 4(q - 1) - 1$. Since Q' separates, p is even, so $ps = 2$ and $s = 1$. Since $e' < ps$ is even, $e' = 0$ and every edge is internal. Viewed in M, $Q' - \eta(k)$ is then an annulus whose boundary components each intersect a meridian of k once. Hence k lies on a torus in M in which Q' is the complement of $\eta(k)$. Finally, a bounded component of $D - \Gamma$ is a compressing disk for the torus. Since $M - k$ is assumed irreducible the torus must bound a solid torus. This is conclusion (c). ‖

§4. SURGERY OF KNOTS IN SUTURED MANIFOLDS

Throughout this section, let (M, γ) be a sutured 3-manifold containing a knot k. As in §3, let M' be a manifold obtained by a Dehn surgery on k with slope $r/s \in \mathbf{Q}/\mathbf{Z}$, and let k' be the core of the filling torus in M'. Assume $(M - \eta(k), \gamma)$ is taut or, equivalently, (M, γ) is k-taut ((M', γ') is k'-taut).

4.1 Proposition. *Let Q' be a sphere or proper disk in M' so that*

(i) $|Q' \cap k'| = p > 0$

(ii) *M contains no Gabai disk for Q'*

(iii) *$Q' - \eta(k')$ is incompressible and ∂-incompressible in $M' - \eta(k')$.*

Then for any k-taut sutured manifold hierarchy

$$(M, \gamma) \xrightarrow{S_1} (M_1, \gamma_1) \rightarrow \ldots \xrightarrow{S_n} (M_n, \gamma_n)$$

for (M, γ) respecting the parameterizing surface Q', either

(a) *the hierarchy is also ϕ-taut*

(b) *M contains a reducing sphere which is disjoint from all the decomposing surfaces S_i and each component of M_n has boundary a single sphere containing exactly one suture or*

(c) *$M = S^1 \times D^2$, Q' is a disk, and k has winding number $w(k) \neq 0$ in M.*

Proof. Apply the argument of [13, 9.1] using the assumption that M contains no Gabai disks wherever [13, 9.3] was used. Indeed [13, 9.1 claims 1–10] shows that, after cancellation and amalgamation of arcs of k in M_n, all ends of k lie on a single spherical boundary component N of ∂M_n and it contains a single suture. The k-tautness of M_n implies that all other components of ∂M_n (which are now disjoint from k) are spheres with a single suture and bound balls.

If the component V of M_n bounded by N is not ϕ-taut then, since the disks $R_\pm$ clearly are taut, V must be reducible. Such a reducing sphere is also a reducing sphere for M and is clearly disjoint from all the S_i.

If V is ϕ-taut then, by 2.7, either every decomposition in the hierarchy is ϕ-taut, yielding (a), or some component of M is a solid torus containing no sutures. In the latter case, since ∂M is incompressible in the complement of k, the solid torus must contain k and so we may assume it's all of M. The surface S_1 in the first decomposition in the hierarchy is then a conditioned surface (2.1a) (hence its boundary is a collection of meridia of the solid torus M) so the manifold (M_1, γ_1) obtained after the decomposition has sutures in each component. Then 2.7 implies (M_1, γ_1) is ϕ-taut as well as k-taut. Then S_1 is a disk, for otherwise it would be compressible in M, forcing $R(\gamma_1)$ to be compressible in M_1 contradicting ϕ-tautness. Finally, if Q' were a sphere or $w(k) = 0$ then $\partial Q' \cap S_1 = \phi$ and S_1 would be a Gabai disk for Q', contradicting (ii). Hence if V is ϕ-taut, (a) or (c) applies. ||

4.2 PROPOSITION. *Suppose M' has a reducing sphere disjoint from every surface in some k'-taut sutured manifold hierarchy for M'. Then there is a reducing sphere Q' with $Q' - \eta(k')$ incompressible and ∂-incompressible in $M' - \eta(k')$ so that either*

(a) *there is no Gabai disk in M for Q' or*
(b) *k is cabled and the slope of the surgery is that of the cabling annulus.*

Proof. If M' contains a sphere whose fundamental class is non-trivial in $H_2(M')$, let Q' be one of minimal intersection with k'. Then 3.3 implies there is no Gabai disk for Q' in M, conclusion (a). So henceforth assume all reducing spheres in M' are null-homologous, hence in particular separating.

Let (M'_m, γ_m) denote the final stage in the given hierarchy of M. Since M'_m contains a reducing sphere for M', some component N of $\partial M'_m$ is a reducing sphere for M'. Let Q' be a reducing sphere for M' chosen so that

(i) Q' is disjoint from the surfaces of the given hierarchy of M'
(ii) no subarc of k' in the region Ω between N and Q' has both ends on Q' and
(iii) subject to conditions (i) and (ii) $|k' \cap Q'| = p$ is minimized. [Note that a push-off of N into M'_m satisfies (i) and (ii).] Q' divides M' into two components; let X denote that containing N and Y denote the other.

We will assume M contains a Gabai disk for Q' and deduce (b). Let D be a Gabai disk chosen so that

(iv) $|Q' \cap D|$ is minimized

and use the terminology of 3.1.

We can assume Γ contains no trivial loop bounding a disk in Y, for such a disk could be used to lower p by an isotopy of k'. Similarly a trivial circle in Γ bounding a disk in Y can be eliminated by a disk-swap with Q'. The same two arguments show that Y cannot contain a compressing or ∂-compressing disk for $Q' - \eta(k')$.

So by 3.4 either k is cabled, conclusion (b), or some bounded component of $D - \Gamma$ lies in X. We henceforth assume the latter and derive a contradiction. An innermost such component, if not simply-connected, cuts off from D a Gabai disk, contradicting (iv). So we can further assume there is a bounded disk component D' of $D - \Gamma$ in X.

Case 1. $\partial D' \subset Q'$

If $\partial D'$ were inessential in $Q' - \eta(k')$ then a disk-swap would produce a Gabai disk with lower $|Q' \cap D|$. Hence by (iv) we can assume both disks of Q' bounded by $\partial D'$ intersect k', so D' compresses $Q' - \eta(k')$ in X. We will show this is impossible (confirming thereby also that $Q' - \eta(k')$ is incompressible in M'). We can alter interior (D') so that it is disjoint from every decomposing surface S_i of the hierarchy by doing disk swaps in each successive decomposing surface that intersects D'. Then D' lies in the region Ω between Q' and N in M'. Since every sphere in M' is null-homologous, D' is homologous rel ∂ in Ω to a subdisk F of Q' which we already know intersects k'. By (ii) there are proper arcs in Ω with one end on F yet disjoint from D', which is absurd since D' and F are homologous.

Case 2. $\partial D'$ does not lie entirely in Q'.

Then $\partial D'$ is the union of arcs alternately lying in Q' (call these arcs $\alpha_1, \ldots, \alpha_n$) and on meridia of k (call these arcs $\beta_1, \ldots, \beta_n$). Consider now the intersections of the decomposing surfaces S_i of the hierarchy with the disk D'. If the first decomposing surface to intersect D' intersects it only in simple closed curves, then a disk swap replaces D' with another disk disjoint from the decomposing surface and equal to D' near its boundary. We know from condition (ii) that eventually some decomposing surface intersects each arc β_i (which run parallel to intervals of k' in M'). Combining these two facts, we may alter the interior of D' so that the first decomposing surface S_i to intersect D' intersects it in a 1-manifold J with $\partial J \neq \phi$. Since S_i is k'-taut it must intersect k' always with the same sign. Since D' came from a Gabai disk, 3.2(a) implies each arc $\beta_i \subset \partial D'$ is oriented by k' in the same direction. But then the orientation inherited by J from D' and S_i must induce the same sign on each end of $\partial J = \partial D' \cap S_i$, which is absurd. The same argument shows that X cannot contain a ∂-compressing disk for $Q' - \eta(k')$, confirming also that $Q' - \eta(k')$ is ∂-incompressible in $M' - \eta(k')$. ||

4.3 THEOREM. *If M' is reducible and either $R(\gamma)$ is compressible or M contains an admissible sphere, then k is cabled and the slope of the surgery is that of the cabling annulus.*

Proof. Choose an admissible sphere E or compressing disk $(E, \partial E) \subset (M, R(\gamma))$ having minimal possible intersection with k. Then apply 3.3 and 4.1, reversing the roles of M and M' and using E for Q': M' contains no Gabai disk for E and for any k'-taut sutured manifold hierarchy of (M', γ') respecting E, M' contains a reducing sphere disjoint from the hierarchy. Moreover each boundary component of M'_m contains exactly one suture. (Note (M', γ') cannot be ϕ-taut nor $M' = S^1 \times D^2$ since M' is assumed reducible). Hence

($*$) (M', γ') satisfies the hypotheses of 4.2.

If ∂M contains a special torus then take for E above a k-taut disk. But E is then a Gabai disk for every reducing sphere in M' so 4.2 implies k is cabled with surgery slope that of the cabling annulus.

If ∂M contains no special torus, then E must be admissible. Construct, as in 2.5, a k-taut manifold hierarchy for M so that (M_n, γ_n) contains an admissible disk or sphere. Then M_n does not satisfy conclusion 4.1(b). Indeed, if there were only one boundary component per component of M_n, then $H_2(M_n) \approx H_2(M_n, \partial M_n) = 0$ (hence there would be no admissible spheres) and if there were only one suture per boundary component then $R(\gamma_n)$ would be all disks (so there would be no admissible disks). Since (M, γ) contains an admissible disk or sphere it is not ϕ-taut so does not satisfy conclusion 4.1(a). Thus by 4.1 we conclude M contains a Gabai disk for any reducing sphere Q' in M' such that $Q' - \eta(k')$ is incompressible and ∂-incompressible in $M' - \eta(k')$. But from ($*$) above such a reducing sphere can be found satisfying the hypotheses of 4.2 and we have just ruled out conclusion 4.2(a). Hence 4.2(b) holds. ||

4.4 COROLLARY. *Suppose k is a knot in a solid torus M, k does not lie in a ball and surgery on k yields a reducible manifold M'. Then k is cabled and the slope of the surgery is that of the cabling annulus.*

Proof. Regard M as a sutured manifold with no sutures. Since ∂M is incompressible in $M - k$, it is k-taut. ||

4.5 COROLLARY. *If surgery on a satellite knot k in S^3 yields a reducible 3-manifold then k is cabled.*

Proof. From [9] we know that the slope of the surgery on k must be integral, say n. We can assume $n \neq 0$ by [5].The result of surgery is a manifold W with $|H_1(W)| = n$.

Let T be the companion torus and M the solid torus in S^3 it bounds. If the manifold M' obtained from M by surgery on k is reducible the conclusion follows from the theorem. If M' is irreducible then an innermost disk argument shows T must be compressible in M'. Since M' is irreducible, it is then a solid torus.

But suppose M were a solid torus. Replacing M with M' can itself be viewed as a Dehn surgery in S^3 on the core of M. The slope of the surgery can be calculated to be n/ω^2, where ω is the winding number of k in M. Since n/ω^2 must be integral we deduce that $|H_1(W)| = n/\omega^2 = n$, so $\omega = 1$. But by [3] k is braided in M (i.e. winds monotonically in $S^1 \times D^2 = M$). But then k must be the core of M, and so is not a satellite. ||

§5. PUTTING SUTURES ON MANIFOLD BOUNDARIES

Theorem 4.3 above, while the central theorem of this paper, is difficult to apply because it is a theorem about sutured manifolds. In this section we show how to apply it to a general 3-manifold, by constructing, in a fairly *ad hoc* fashion, a collection of sutures making the resulting 3-manifold taut. The difficulty is to ensure that there are no surfaces in the interior of the 3-manifold with small Thurston norm; the strategy is to place the sutures so that the smallest Thurston norm possible is that which the sutures already bound in the boundary.

5.1 *Definitions.* Two disjoint simple closed curves c and c' on the boundary of a 3-manifold M are *co-annular* if they bound a properly imbedded annulus A in M. Let S be a connected orientable surface of genus $g > 1$. A collection of $3g - 3$ disjoint simple closed curves S is *standard* if each complementary component is a pair of pants, $g - 1$ of the curves are separating and $2g - 2$ are non-separating. (See Fig. 5.1). A collection of disjoint simple closed curves on an orientable surface is standard if it is standard on each component. A collection of curves is *substandard* if it is contained in some standard collection.

Let M be a ∂-irreducible compact orientable 3-manifold and $\xi \subset \partial M$ be a substandard collection of simple closed curves.

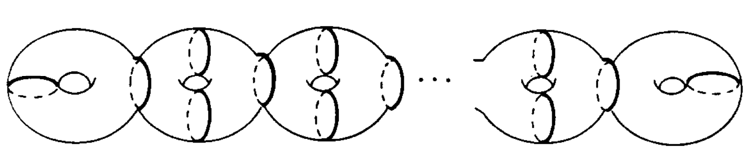

Fig. 5.1.

5.2 LEMMA. *ξ is contained in a standard collection Ξ so that no curve in Ξ-ξ is coannular in M to a curve in Ξ.*

Proof. The proof is by induction on $|\Xi - \xi|$.

If ξ is standard there's nothing to prove. If not, there is a component E of $\partial M - \xi$ which is not a pair of pants. Contained in E is a curve c so that $\xi \cup \{c\}$ is also substandard. If c is not co-annular with any curve in ξ then replacing ξ by $\xi \cup \{c\}$ completes the inductive step. So assume an annulus A in M has boundary $c \cup \alpha$, where $\alpha \in \xi$.

If c is non-separating in E, there is a c' in E intersecting E in a single point. Suppose A' were an annulus with boundary $c' \cup \beta$, $\beta \in \xi$. A small isotopy moves A and A' into general position and makes α and β disjoint. $A \cap A'$ would then be a compact 1-manifold with a single end, which is absurd. Thus c' can't be coannular with a curve in ξ. There is an automorphism of E, preserving ∂E, which carries c to c', so $\xi \cup \{c'\}$ is also substandard. So replacing ξ by $\xi \cup \{c'\}$ completes the inductive step.

If c separates E then c lies in a 4-punctured sphere E' contained in E with $\partial E'$ essential in ∂M and two components of $\partial E'$ lying on each side of c. Consider the curve c' shown in Fig. 5.2.

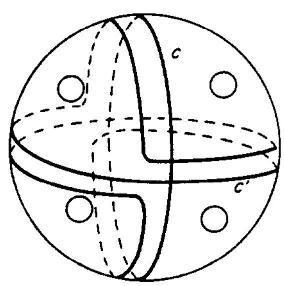

Fig. 5.2.

Suppose there is a proper annulus A' with $\partial A' = c' \cup \beta$, β in ξ. After a small isotopy putting A and A' in general position, $\partial A \cap \partial A'$ consists of the four points of $c \cap c'$ and $A \cap A'$ contains two arcs with these as end-points. Each of the arcs cuts off a disk from A and A'. The union of two such disks along an arc of $A \cap A'$ would be a compressing disk for ∂M, a contradiction. Thus c' can't be coannular with a curve in ξ. There is an automorphism of E, preserving ∂E, which carries c to c' (indeed a single Dehn twist suffices) so $\xi \cup \{c'\}$ is also substandard. Replacing ξ by $\xi \cup \{c'\}$ completes the inductive step. ||

5.3 *Definition.* A standard collection of curves on ∂M such that no pair of curves is coannular in M is called a *special collection.*

5.4 COROLLARY. *Any curve (pair of non-coannular curves) in ∂M is contained in a special collection.*

Proof. Clearly any curve or pair curves which are non-co-annular (hence non-parallel in ∂M) is substandard. ||

5.5 *Definition*: A collection of curves on ∂M which is contained in some special collection is called *subspecial.* A subspecial collection of curves Ξ on ∂M is *pantsless* if whenever three elements of Ξ bound a properly imbedded pair of pants in M, all three lie on the same component of ∂M.

5.6 LEMMA. *Let M be a ∂-irreducible compact orientable 3-manifold and $\xi \subset M$ be a pantsless subspecial collection of simple closed curves. Then ξ is contained in a pantsless special collection.*

Proof. We will show how to expand ξ to a larger pantsless subspecial collection. Repeating the argument sufficiently often yields the lemma.

Let Ξ be a special collection containing ξ, and E a component of $\partial M - \xi$ not a pair of pants. Then E contains a curve c of $\Xi - \xi$. c is contained either in a once-punctured torus or a 4-punctured sphere E' contained in E so that E' is disjoint from any other curve in Ξ, each component of $\partial E'$ is parallel in E to a curve in Ξ, and any two components of $\partial E'$ parallel to the same curve in Ξ are separated in E' by c (see e.g. Fig. 5.3 in the case c is non-separating). We suppose $\xi \cup \{c\}$ is not pantsless, i.e. there is a proper pair of pants P in M with $\partial P = c \cup \alpha_1 \cup \alpha_2$, so that α_1 and α_2 are in ξ and α_2, say, lies in a different component than the component S of ∂M containing c.

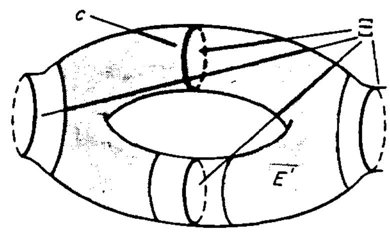

Fig. 5.3.

Case 1. E' is a once-punctured torus. Let c' be the curve intersecting c in a single point. Suppose there is an annulus A' in M with $\partial A' = c' \cup \beta$, β in Ξ. Then after isotoping A' and P to minimize their intersection we discover $|A' \cap P|$ is a compact 1-manifold with a single end, which is absurd. Hence the collection Ξ' obtained from Ξ by replacing c with c' remains special. Similarly if there is a pair of pants P' in M with $\partial P' = c' \cup \beta_1 \cup \beta_2$, with β_1, β_2 in ξ. Hence $\xi \cup \{c'\}$ is pantsless and subspecial.

Case 2. E' is a 4-punctured sphere. Let c' be the curve shown in Fig. 5 .2, Ξ' the collection obtained from Ξ by replacing c with c'.

Claim 1. Ξ' is special.

Proof of Claim 1. Suppose there is an annulus A' in M with $\partial A' = c' \cup \beta$, β in Ξ. Then after isotoping A' and P to minimize their intersection, $|A' \cap P|$ is a compact 1-manifold with the four end-points $c \cap c'$. After perhaps some disk and annulus swaps we may alter A' so that $|A' \cap P|$ is a pair of arcs, each of which must cut off a disk from A'. One of the arcs cuts off from P either a disk, or an annulus whose other boundary component is α_2. The union of the disk in A' and the disk (or annulus) in P is a disk (annulus) with boundary (one end) a component of $\partial E'$ (and the other α_2, which is not in S). This is impossible since M is ∂-irreducible (Ξ is special). The contradiction proves the claim.

Claim 2. $\xi \cup \{c'\}$ is pantsless.

Proof of Claim 2. Suppose P' is a proper pair of pants in M with $\partial P' = c' \cup \beta_1 \cup \beta_2$, with β_1, β_2 in ξ and β_2 not in S. After a small isotopy, $P \cap P'$ is a compact 1-manifold with ends the four points of $c \cap c'$, so $P \cap P'$ is a union of two arcs and some circles. Since ∂M is incompressible, the interiors of P and P' may be altered by disk-swaps to eliminate any inessential circles in $P \cap P'$. Suppose there is a circle of intersection parallel to β_2 in P'. The corresponding circle in P must be parallel to α_2 in P since Ξ is special, so an annulus swap alters P, replacing α_2 with β_2 and lowers $|P \cap P'|$. Continue until no circle of intersection is parallel to β_2. Since Ξ' is also special we may similarly eliminate any circles parallel in P to α_2, and then proceed to eliminate circles of intersection parallel to α_1 and β_1. If there are any circles of intersection left, they must be parallel to c and c', which forces the arcs of $|P \cap P'|$ to cut off disks from P and P'. As above, the union of two disks along an arc would produce a ∂-reducing disk in M, which is impossible. Hence we may assume $|P \cap P'|$ consists precisely of two arcs γ_1 and γ_2.

Suppose one of the arcs cuts off a disk in P or P'. Say γ_1 is an outermost arc cutting off a disk from P. Then γ_1 cuts off from P' either a disk or an annulus containing β_2. Then the union along γ_1 of the disk in P and the disk (annulus) in P' is a disk (annulus) which contradicts either the ∂-irreducibility of M or the assumption that Ξ is special. Thus we can assume that no γ_i cuts off a disk from either P or P', so γ_1 is parallel to γ_2 in both P and P'.

The union of the rectangle R lying between γ_1 and γ_2 in P and the rectangle R' between tem in P' is an annulus whose boundary consists of two components of $\partial E'$ lying on the same side of c, hence parallel to distinct curves in Ξ. This contradicts the assumption that Ξ is special. The contradiction proves the claim, and with it, 5.6. ||

5.7. *Definition* In any standard collection of curves in a connected oriented surface S of genus $g > 1$ there are two curves each of which lie on the boundary of a single pair of pants. Call these *redundant* (see Fig. 5.4). Suppose Ξ is a standard collection of curves on ∂M. Let γ be Ξ with all redundant curves removed and label the complementary regions of Γ in S alternately R_+ and R_- as shown in Fig. 5.4. Assign any toral components of ∂M to either R_+ or R_-. Say that the resulting sutured manifold structure (M, γ) on M is *associated* with Ξ.

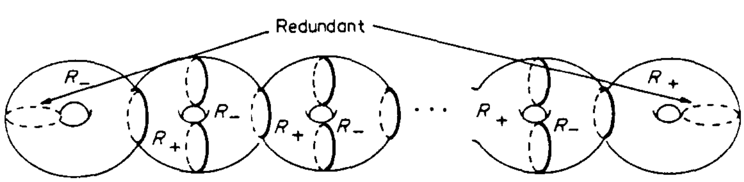

Fig. 5.4.

5.8 PROPOSITION. *Let M be an irreducible, ∂-irreducible compact orientable 3-manifold.*

(a) *If ∂M has a unique non-toral component S, then any satured manifold structure (M, γ) associated to a special collection of curves in S is taut.*

(b) *If ∂M has two non-toral components S and S', then any sutured manifold structure (M, γ) associated to a pantsless collection of curves in $S \cup S'$ is taut.*

Proof. (a) Let $g = \text{genus } S$ and note that $\chi_-(R_+) = \chi_-(R_-) = g - 1$ and $|\gamma| = 3g - 5$. It suffices to show that any oriented surface T in M with $\partial T = \gamma$ has $\chi_-(T) \geq g - 1$. Since M is ∂-irreducible no component of T is a disk and we may assume none is a sphere. Since Ξ is special T contains no annuli. If T' is a component of T with genus $(T') \geq 1$ then $\chi_-(T') = -\chi(T') = 2(\text{genus } T') - 2 + |\partial T'| \geq |\partial T'|$. If genus $(T') = 0$ then $|\partial T'| \geq 3$ so $\chi_-(T') = -2 + |\partial T'| \geq |\partial T'|/3$, with equality only if $|\partial T'| = 3$. Hence $\chi_-(T) \geq |\partial T|/3 = |\gamma|/3 = (3g - 5)/3$. Since $\chi_-(T)$ is an integer, we conclude $\chi_-(T) \geq g - 1$ as required.

(b) Let g denote the genus of $S \cup S'$ and note that $\chi_-(R_+) = \chi_-(R_-) = g - 2$ and $|\gamma| = 3g - 10$. Suppose T is an oriented surface in M with $\partial T = \gamma$.

If each component of T has its entire boundary on the same component of ∂M, the proof is exactly as in (a). So suppose some component T_0 of T has part of its boundary on S and part on S'. If T_0 is planar, then since Ξ is pantsless, $|\partial T_0| \geq 4$ and $\chi_-(T_0) = |\partial T_0| - 2$. If T_0 is not planar then $\chi_-(T_0) \geq |\partial T_0|$. Hence in general $\chi_-(T_0) \geq (|\partial T_0| + 2)/3$. The argument of (a) above shows that $\chi_-(T - T_0) \geq |\partial(T - T_0)|/3$. Hence $\chi_-(T) \geq (|\partial T| + 2)/3 = (3g - 8)/3 = g - 8/3$. Since $\chi_-(T)$ is in fact an integer, $\chi_-(T) \geq g - 2$, as required. ||

5.9 PROPOSITION. *Let M be an irreducible, ∂-irreducible compact orientable 3-manifold and suppose ∂M contains at most two non-toral boundary components.*

(i) *If c is an essential simple closed curve in ∂M then there is a taut sutured manifold structure on M for which c lies in $R(\gamma)$.*

(ii) *If c and c' are disjoint essential, simple closed curves on ∂M which are non-co-annular in M then there is a taut sutured manifold structure on M for which both c and c' lie in $R(\gamma)$.*

Proof. We prove (ii), since (i) is similar but easier. By 5.4 the pair $\{c, c'\}$ is subspecial; since there are only two elements it must then be pantsless. By 5.6 there is a pantsless special collection Ξ in ∂M disjoint from c and c'. Indeed, there is a pantsless special collection containing whichever of c and/or c' lie on non-toral boundary components. Push c and c' slightly off Ξ. By 5.8 a sutured manifold structure associated to Ξ is taut. ||

§6. PROOF OF THE MAIN THEOREM.

In this section we combine results from §4 and §5 to produce a proof of

6.1 THEOREM. *Let M be a compact orientable 3-manifold. Suppose k is a knot in M with $M - k$ irreducible and ∂-irreducible. Let M' be a manifold obtained by Dehn surgery on M, with $k' \subset M'$ the core of the filling torus. If ∂M compresses in M or M contains a sphere not bounding a rational homology ball then either·*

(a) *$M' = D^2 \times S^1 = M$ and both k and k' are 0 or 1-bridge braids*

(b) *$M' = D^2 \times S^1$, $M = D^2 \times S^1 \# L$ for some Lens space L, k is the knot sum of the core of L and a 0-bridge braid, and k' is the cable on a 0-bridge braid.*

(c) *k is cabled and the slope of the surgery is that of the cabling annulus or*

(d) *M' is irreducible. No torus component of ∂M compresses in both M and M'. Any pair of simple closed curves c; $c' \subset \partial M$ which compress in M and M' respectively must intersect. In fact, if M contains a sphere not bounding a rational homology ball, M' is ∂-irreducible.*

Proof.

Case 1. $M - \eta(k)$ is an i-cobordism (cf. [4]). In particular ∂M is a torus.

In this case, apply 4.3 to M and M', regarding M as a k-taut sutured manifold with $\gamma = \phi$. The hypothessis implies that M contains a compressing disk for $R(\gamma)$ or an admissible sphere. If M' is reducible then 4.3 implies (c). If M' is irreducible, ∂-irreducible, this is case (d). If M' is irreducible but ∂-reducible then M' is a solid torus. If M is irreducible then also $M = S^1 \times D^2$ and [3] shows that k and k' are both 0 or 1-bridge braids. This is case (a).

So finally suppose M' is a solid torus and M is reducible. From 4.3 reversing the roles of M and M': k' is cabled with surgery the slope of the cabling annulus. This implies

(i) $M = W \# L$, L a Lens space

(ii) k is the knot sum of a knot l in W and a core of L

(iii) W can be obtained from M' by surgery on the core $l' \subset M'$ of the solid torus on which k' is cabled. The slope is that of k' in the solid torus: $p/q \in \mathbf{Q}/\mathbf{Z}$ where $q \geq 2$ since k' is not isotopic to l'.

Since M was ∂-reducible, so is W. Repeat the above argument on W and M'. This time we know that the surgery slope on l' is not that of a cabling annulus, since a cabling annulus always has integral slope. Hence W must be irreducible, i.e. $S^1 \times D^2$. By [3] l and l' are then 0 or 1-bridge braids. Furthermore, if either is a 1-bridge braid then the surgery slope is integral [6, 3.2] on both. We conclude that l and l' are 0-bridge braids.

Case 2. $M - \eta(k)$ is not an i-cobordism, M' is reducible.

M either contains a reducing sphere or there is a simple closed curve $c \subset \partial M$ which compresses in M. We will assume the latter; the former case is similar. Let S be the component of ∂M on which c lies.

If every other component of ∂M is a torus then by 5.9 there is a taut sutured manifold structure on $M - \eta(k)$ so that c lies in $R(\gamma)$. Case (c) then follows from 4.3.

Otherwise, attach to each non-toral component T of $\partial M - S$ an irreducible, ∂-irreducible, atoroidal and an-annular 3-manifold W_T with $\partial W_T \cong T$ and $H_2(W_T) \neq 0$ (see [12]). c continues to compress in $\tilde{M}$, $\tilde{M}'$ remains reducible, and $\tilde{M} - \eta(k)$ remains irreducible and ∂-irreducible. (Also a sphere in M not bounding a rational homology ball will not bound one in $\tilde{M}$ since each $H_2(W_T) \neq 0$.) Apply the above argument to $\tilde{M}$ and $\tilde{M}'$ and deduce that k is cabled in $\tilde{M}$ with surgery the slope of the cabling annulus A. Consider $\partial M \cap A$. We can assume no component is inessential in A since $M - \eta(k)$ and each W_T is ∂-irreducible. We can assume no component of $A - \partial M$ is an annulus lying outside M since W_T is an-annular. Hence A is disjoint from ∂M. Since T is incompressible in $\tilde{M} - \eta(k)$, it can't lie inside the solid torus on which k is cabled.Thus k is in fact cabled in M, conclusion (c).

Case 3. $M - \eta(k)$ is not an i-cobordism, M' is irreducible.

We will suppose $c' \subset \partial M'$ is a simple closed curve compressing in M' and derive a contradiction if either $c \subset \partial M \doteqdot \partial M'$ is a disjoint simple closed curve compressing in M or M contains a sphere not bounding a rational homology ball. In fact we will suppose the former: the contradiction in the latter case would be derived similarly. Let S and S' be the components of ∂M on which c and c' respectively lie. If c and c' are co-annular in $M - \eta(k)$ then c compresses in both M and M'. Hence we may assume that either $c = c'$ or that c and c' are not co-annular in $M - \eta(k)$.

Subcase (i). All components of ∂M other than S, S' are tori.

Then by 5.9 there is a taut sutured manifold structure on $M - \eta(k)$ and c and c' lie in $R(\gamma)$. Since c and c' compress in M and M' respectively, the structure is not taut after k and k' are filled in. There is a separating torus T_1 in $M - \eta(k)$ for which the component N of $M - (\eta(k) \cup T_1)$ containing $T_0 = \dot{\eta}(k)$ is an i-cobordism between T_0 and T_1 (see [4]). Moreover, N has a taut sutured manifold structure with a non-empty set of sutures γ, all of which lie on T_1, but after filling in $\eta(k)$ and $\eta(k')$, N is no longer taut.

Since N is an i-cobordism, only one of the two fillings makes $R_{\pm}$ compressible. Hence the other makes N reducible. Since M', hence $N \cup \eta(k')$, is irreducible, $N \cup \eta(k)$ is reducible. Furthermore, an innermost disk argument shows that T_1 is compressible after both fillings. Then by 4.3, k' is cabled and the surgery on k' is that of the cabling annulus. As in case (i) this implies that

(i) $M = W \# L$, L a Lens space

(ii) k is the knot sum of a knot l in W and a core of L

(iii) W can be obtained from M' by surgery on the core l' of the solid torus on which k' is cabled. The slope is that of k' in the solid torus: $p/q \in \mathbf{Q}/\mathbf{Z}$ where $q \geq 2$.

c and c' are still compressible in W and M' respectively, so repeat the argument above to obtain that l' is cabled, and the slope of the surgery is that of the cabling annulus, hence integral. This contradicts conclusion (iii) that $q \geq 2$.

Subcase (ii): $\partial M - (S \cup S')$ contains components not tori.

Construct $\tilde{M}$ and $\tilde{M}'$ as in case 2, attaching to each non-toral component of $\partial M - (S \cup S')$ an irreducible, ∂-irreducible, atoroidal and an-annular 3-manifold W_T with $\partial W_T \cong T$ and $H_2(W_T) \neq 0$. c and c' still compress in $\tilde{M}$ and $\tilde{M}'$, $\tilde{M}'$ remains irreducible, and $\tilde{M} - \eta(k)$ remains irreducible and ∂-irreducible. Then the argument of Subcase (i) applied to $\tilde{M}$ and $\tilde{M}'$ again yields a contradiction.

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Added in proof: Y.-Q. Wu has pointed out the following omission. In the Proof of 4.3, when 4.1 is applied with the roles of M and M' reversed, it is tacitly assumed that M' has a sutured manifold hierarchy, i.e. that $H_1(M'; Q) \neq 0$. If, in fact, $H_1(M'; Q) = 0$ then $\beta_1(M) \leq 1$, so any 2-sphere in M not bounding a rational homology ball must be non-separating. This case is not covered by 4.3 and so should be added to 8.1:

(e) $M' = (S^1 \times S^2) \# W$, $M' = W_1 \# W_2$, where W, W_1 and W_2 are rational homology spheres.

We may further conclude, using the techniques above, that either $H_1(W) \neq 0$ or W_1 is a Lens space. *This correction does not affect the main applications, 4.4 and 4.5.*

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