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EQUIVALENCE OF 5-DIMENSIONAL s-COBORDISMS

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ABSTRACT. The classification of 5-dimensional h-cobordisms given by Cappell, Lashof, and Shaneson is here strengthened and extended to s-cobordisms when the ends of the s-cobordism are smooth.

1. **Introduction.** An s-cobordism between compact manifolds M and M' will be an s-cobordism which restricts to a product cobordism between ∂M and $\partial M'$. Two s-cobordisms W and W' from a compact 4-manifold to a smooth manifold are equivalent if there are smooth s-cobordisms V and V' with $\partial_2 W = \partial_1 V$, $\partial_2 W' = \partial_1 V'$ and a homeomorphism of $W \cup V$ onto $W' \cup V'$ which is the identity on $M = \partial_1 W$ and a diffeomorphism from $\partial_2 V$ to $\partial_2 V'$. Given a smooth 4-manifold M, let M_k denote the connected sum of M and k copies of $S^2 \times S^2$.

Theorem. *There is a k such that for any connected compact smooth 4-manifold M there is a 1-1 correspondence between $H^3(M, \partial M; Z_2)$ and equivalence classes of s-cobordisms of M_k to a smooth manifold.*

The correspondence θ is defined as follows: Given a representative W of an equivalence class $[W]$ of s-cobordisms there is an obstruction α in $H^4(W, \partial W; Z_2)$ to extending the smooth structure on ∂W to all of W [2]. The exact cohomology sequence for the triple $(W, \partial W, \partial W - M_k)$ provides a natural isomorphism $\delta : H^3(\partial W, \partial W - M_k; Z_2) \to H^4(W, \partial W; Z_2)$. There is also an excision isomorphism $e : H^3(\partial W, \partial W - M_k; Z_2) \to H^3(M_k, \partial M_k; Z_2)$ and a "projection" isomorphism $p^*: H^3(M, \partial M; Z_2) \to H^3(M_k, \partial M_k; Z_2)$. Set $\theta([W]) = p^{*-1} e \delta^{-1}(\alpha)$.

A similar theorem is proven for h-cobordisms of closed topological 4-manifolds in [1]. There k depends on M, here it does not. In fact, if M is orientable we may take $k = 1$.

2. **Proof of the Theorem.** We require the following

Lemma. *Let $(W; M, M')$ be a* TOP *s-cobordism between compact 4-manifolds M and M'. Then there is a homeomorphism $W \cup_{M'} W \to M \times I$.*

Proof of the Lemma. The manifold $W \times I$ is a TOP s-cobordism from $W \cup_{M'} W$ to $M \times I$. The Lemma then follows from the high dimensional TOP s-cobordism theorem [3].

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Case 1. *For any k, θ is injective.*

Proof of Case 1. Suppose W' and W'' are TOP s-cobordisms from M_k to smooth manifolds M' and M'', respectively, and $\theta([W']) = \theta([W''])$. It follows from the definition of θ that $W' \cup_{M_k} W''$ is a smooth s-cobordism from M' to M''. Therefore W' and W'' are equivalent s-cobordisms, for, by the Lemma,

$$W' \cup_{M'} (W' \cup_{M_k} W'') = (W' \cup_{M'} W') \cup_{M_k} W'' = (M_k \times I) \cup_{M_k} W'' = W''.$$

This proves Case 1.

Case 2. *M is a 3-disk bundle over S^1.*

Proof of Case 2. By Case 1, we need only find a k such that θ is onto. The double, $2M$, of M is a 3-sphere bundle over S^1. There is an integer j, depending on M, and a topological h-cobordism H from $(2M)_j$ to a smooth manifold $(2M)'$ such that the natural smoothing near ∂H fails to extend to all of H [1]. Let T and T' be smoothly imbedded circles in $(2M)_j$ and $(2M)'$, respectively, which represent a generator of $\pi_1(H) = Z$. By general position T and T' are concordant. Remove an open tubular neighborhood ν of the concordance, chosen so that $(2M)_j - \nu = M_j$.

Standard arguments now show that the resulting manifold G is an s-cobordism from M_j to a smooth manifold and the natural smoothing of G does not extend to all of G. Hence $\theta([W])$ is the nontrivial element of $H^3(M, \partial M; Z_2) = Z_2$, so θ is onto.

There are only two 3-disk bundles over S^1, yielding two values for j. The proof is completed by letting k be the larger.

Case 3. *General case, k as in Case 2.*

Proof of Case 3. By Case 1 it suffices to show that θ is onto. Given α in $H^3(M, \partial M; Z_2)$ let S be a smoothly imbedded circle in M representing the Poincaré dual to α in $H_1(M; Z_2)$, and let $\nu(S)$ be a tubular neighborhood of S. Replace $\nu(S) \times I$ in $M \times I$ by the s-cobordism G defined in Case 2 for the disk bundle $\nu(S)$. The result is a topological s-cobordism W from M_k to a smooth manifold. It is easily seen that $\theta([W]) = \alpha$.

3. **Remarks.** It is shown in [4] that when M is the orientable disk bundle over S^1 we may take $k = 1$. Hence, in general, whenever M is an orientable manifold, we may take $k = 1$.

When M is closed, the correspondence θ^{-1} coincides with the correspondence defined in [1] up to h-cobordism.

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