SURFACES, SUBMANIFOLDS, AND ALIGNED FOX REIMBEDDING IN NON-HAKEN 3-MANIFOLDS

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Abstract. Understanding non-Haken 3-manifolds is central to many current endeavors in 3-manifold topology. We describe some results for closed orientable surfaces in non-Haken manifolds, and extend Fox’s theorem for submanifolds of the 3-sphere to submanifolds of general non-Haken manifolds. In the case where the submanifold has connected boundary, we show also that the $\partial$-connected sum decomposition of the submanifold can be aligned with such a structure on the submanifold’s complement.

1. Introduction

A closed orientable irreducible 3-manifold $N$ is called Haken if it contains a closed orientable incompressible surface; otherwise $N$ is non-Haken. In Section 2 we describe some results for surfaces in non-Haken manifolds. Generalizing a theorem of Fox ([F]), we show in Section 3 that a 3-dimensional submanifold of a non-Haken manifold $N$ is homeomorphic either to a handlebody complement in $N$ or the complement of a handlebody in $S^3$. Sections 2 and 3 are independent, but both represent progress towards understanding submanifolds of non-Haken manifolds. In Section 4 we combine the techniques from Section 2 with the results from Section 3 to show that if the submanifold $M \subset N$ is $\partial$-reducible and has connected boundary, then the embedding can be chosen to align a full collection of separating $\partial$-reducing disks in $M$ with similar disks in the complement of $M$.

2. Handlebodies in non-Haken manifolds

Let $N$ be a closed orientable 3-manifold, $F$ a closed orientable surface of non-trivial genus imbedded in $N$. Recall that $F$ is compressible if there exists an essential simple closed curve on $F$ which bounds an imbedded disk $D$ in $N$ with interior disjoint from $F$. $D$ is a compressing disk for $F$.

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Definition 1. Suppose $F$ is a separating closed surface in an orientable irreducible closed 3-manifold $N$. $F$ is reducible if there exists an essential simple closed curve on $F$ which bounds compressing disks on both sides of $F$. The union of the two compressing disks is a reducing sphere for $F$.

Suppose $S$ is a collection of disjoint reducing spheres for $F$. A reducing sphere $S' \in S$ is redundant if a component of $F - S$ that is adjacent to $S \cap F$ is planar. $S$ is complete if, for any disjoint reducing sphere $S'$, $S'$ is redundant in $S \cup S'$.

Let $\sigma(S)$ denote the number of components of $F - S$ that are not planar surfaces.

Since $N$ is irreducible, any sphere in $N$ is necessarily separating. Suppose a reducing sphere $S'$ is added to a collection $S$ of disjoint reducing spheres. If $S'$ is redundant, the number of non-planar complementary components in $F$ is unchanged, since $S'$ necessarily separates the component of $F - S$ that it intersects and the union of two planar surfaces along a single boundary component is still planar. If $S'$ is not redundant then the number of non-planar complementary components in $F$ increases by one. Thus we have:

Lemma 2. Suppose $S \subset S'$ are two collections of disjoint reducing spheres for $F$ in $N$. Then $\sigma(S) \leq \sigma(S')$. Equality holds if and only if each sphere $S'$ in $S' - S$ is redundant in $S' \cup S$. In particular, $S$ is complete if and only if for every collection $S'$ such that $S \subset S'$, $\sigma(S) = \sigma(S')$.

Let $H$ be a handlebody imbedded in $N$. $H$ has an unknotted core if there exists a pair of transverse simple closed curves $c, d \subset \partial H$ such that $c \cap d$ is a single point, $d$ bounds an imbedded disk in $H$ and $c$ (the core) bounds an imbedded disk in $N$.

Lemma 3. Let $F$ be a connected, closed, separating, orientable surface in a closed orientable irreducible 3-manifold $N$. Suppose that $F$ has compressing disks to both sides. Then at least one of the following must hold:

1. $F$ is a Heegaard surface for $N$.
2. $N$ is Haken.
3. There exist disjoint compressing disks for $F$ on opposite sides of $F$.

Proof. The proof is an application of the generalized Heegaard decomposition described in [ST]. As $F$ is compressible to both sides, we can
construct a handle decomposition of $N$ starting at $F$ so that $F$ appears as a “thick” surface in the decomposition. If $F$ is not a Heegaard surface, then this decomposition contains a “thin” surface $G$ adjacent to $F$. If $G$ is incompressible in $N$, then $N$ is Haken. If $G$ is compressible we apply [CG] to obtain the required disjoint compressing disks for $F$. \hfill $\square$

**Theorem 4.** Let $H$ be a handlebody of genus $g$ imbedded in a closed orientable irreducible non-Haken 3-manifold $N$. Let $G$ be the complement of $H$ in $N$. Let $F = \partial H = \partial G$. Suppose $F$ is compressible in $G$. Then at least one of the following must hold:

1. The Heegaard genus of $N$ is less than or equal to $g$.
2. $F$ is reducible.
3. $H$ has an unknotted core.

**Proof.** The proof is by induction on the genus of $H$. If $g = 1$, then the result of compressing $F$ into $G$ is a 2-sphere, necessarily bounding a ball in $N$. If a ball it bounds lies in $G$ then the Heegaard genus of $N$ is less than or equal to $1$. If a ball it bounds contains $H$ then $H$ is an unknotted solid torus in $N$ and so it has an unknotted core.

Suppose then that $\text{genus}(H) = g > 1$ and assume inductively that the theorem is true for handlebodies of genus $g - 1$. Suppose that $G$, the complement of $H$, has compressible boundary. If $G$ is a handlebody then $G \cup F H$ is a Heegaard splitting of genus $g$ and we are done. So suppose $G$ is not a handlebody. Then by Lemma 3 there are disjoint compressing disks on opposite sides of $F$, say $D$ in $H$ and $E$ in $G$. Without loss of generality we can assume that $D$ is non-separating. Compress $H$ along $D$ to obtain a new handlebody $H_1$ with boundary $F_1$; let $G_1$ be the complement of $H_1$.

If $\partial E$ is inessential in $F_1$ then it bounds a disk in $H_1 \subset H$ as well, so $F$ is reducible.

If $\partial E$ is essential in $F_1$ then $E$ is a compressing disk in $G_1$ so we can apply the inductive hypothesis to $H_1$. If 1 or 3 holds then it holds for $H$, and we are done. Suppose instead $F_1$ is reducible. Let $S$ be a collection of disjoint reducing spheres for $F_1$ chosen to maximize $\sigma$ among all possible such collections and then, subject to that condition, further choose $S$ to minimize $|E \cap S|$. Clearly $E \cap S$ contains no closed curves, else replacing a subdisk lying in the disk collection $S \cap G_1$ with an innermost disk of $E - S$ would reduce $|E \cap S|$. Similarly, we have

**Claim 1** Suppose $\epsilon$ is an arc component of $\partial E - S$ and $F_0$ is the component of $F_1 - S$ in which $\epsilon$ lies. If $\epsilon$ separates $F_0$ (so the ends of $\epsilon$...
necessarily lie on the same component of \( \partial F_0 \) then neither component of \( F_0 - \epsilon \) is planar.

**Proof of claim 1:** Let \( \alpha_0 \) be the closed curve component of \( \partial F_0 \cap S \cap F_1 \) on which the ends of \( \epsilon \) lie and, of the two arcs into which the ends of \( \epsilon \) divide \( \alpha_0 \), let \( \alpha \) be adjacent to a planar component of \( F_0 - \epsilon \). Then the curve \( \epsilon \cup \alpha \) clearly bounds a disk in both \( G_1 \) and \( H_1 \) and then so does the curve \( \epsilon' = \epsilon \cup (\alpha_0 - \alpha) \). Let \( S' \) be a sphere in \( N \) intersecting \( F_1 \) in \( \epsilon' \) and \( S_0 \) be the reducing sphere in \( S \) containing \( \alpha_0 \). Replacing \( S_0 \) with \( S' \) (or just deleting \( S_0 \) if \( \epsilon' \) is inessential in \( F_1 \)) gives a new collection \( S' \) of disjoint reducing spheres, intersecting \( \partial E \) in at least two fewer points. Moreover \( \sigma(S') = \sigma(S) \) since the only change in the complementary components in \( F_1 \) is to add to one component and delete from another a planar surface along an arc in the boundary. Then the collection \( S' \) contradicts our initial choice for \( S \), a contradiction that proves the claim.

Let \( H' \) be the closed complement of \( S \) in \( H_1 \), so \( H' \) is itself a collection of handlebodies.

**Claim 2** Either \( F \) is reducible or \( \partial H' \) is compressible in \( N - H' \).

**Proof of claim 2:** If \( \partial E \) is disjoint from \( S \) and is inessential in \( \partial H' \), then \( \partial E \) bounds a disk in \( H' \), hence in \( H \), so \( F \) is reducible. If \( \partial E \) is disjoint from \( S \) and is essential in \( \partial H' \), then \( E \) compresses \( \partial H' \) in \( N - H' \), verifying the claim. Finally, if \( E \) intersects \( S \), consider an outermost disk \( A \) cut off from \( E \) by \( S \). According to Claim 1, this disk, together with a subdisk of \( S \), constitute a disk \( E' \) that compresses \( \partial H' \) in \( N - H' \), proving the claim.

Following Claim 2, either \( F \) is reducible or the inductive hypothesis applies to a component \( H_0 \) of \( H' \). If 2 holds for \( H_0 \) then consider a reducing sphere \( S \) for \( H_0 \), isotoped so that the curve \( c = S \cap \partial H_0 \) is disjoint from the disks \( S \cap H_0 \). The disk \( S - H_0 \) may intersect \( H_1 \); by general position with respect to the dual 1-handles, each component of intersection is a disk parallel to a component of \( S \cap H_1 \). But each such disk can be replaced by the corresponding disk in \( S - H_1 \) so that in the end \( c \) also bounds a disk in \( N - H_1 \). After this change, \( S \) is a reducing sphere for \( F_1 \) in \( N \) and, since \( c \) is essential in \( H_0 \), \( \sigma(S \cup S) > \sigma(S) \), contradicting our initial choice for \( S \). Thus in fact 1 or 3 holds for \( H_0 \), hence also for \( H \).

In the specific case \( N = S^3 \), we apply precisely the same argument, combined with Waldhausen's theorem [W] on Heegaard splittings of \( S^3 \), to obtain:
Corollary 5. Let $H$ be a handlebody imbedded in $S^3$, and suppose $G$, the complement of $H$, has compressible boundary. Then either $H$ has an unknotted core or the boundary of $H$ is reducible.

This corollary is similar to ([MT], Theorem 1.1), but no reimbedding of $S^3 - H$ is required.

3. Complements of handlebodies in non-Haken manifolds

In [F] (see also [MT] for a brief version) Fox showed that any compact connected 3-dimensional submanifold $M$ of $S^3$ is homeomorphic to the complement of a union of handlebodies in $S^3$. We generalize this result to non-Haken manifolds, showing that a submanifold $M$ of a non-Haken manifold $N$ has an almost equally simple description, that is, $M$ is homeomorphic to the complement of handlebodies either in $S^3$ or in $N$.

Definition 6. Let $N$ be a compact irreducible 3-manifold, and let $M$ be a compact 3-submanifold of $N$. We will say the complement of $M$ in $N$ is standard if it is homeomorphic to a collection of handlebodies or to $N\#(\text{collection of handlebodies})$. (We regard $B^3$ as a handlebody of genus 0.)

Note that in the latter case $M$ is actually homeomorphic to the complement of a collection of handlebodies in $S^3$.

Theorem 7. Let $N$ be a closed orientable irreducible non-Haken 3-manifold, and let $M$ be a connected compact 3-submanifold of $N$ with non-empty boundary. Then $M$ is homeomorphic to a submanifold of $N$ whose complement is standard.

Proof. The proof will be by induction on $n + g$ where $n$ is the number of components of $\partial M$ and $g$ is the genus of $\partial M$, that is, the sum of the genera of its components. If $n + g = 1$ then $\partial M$ is a single sphere. Since $N$ is irreducible, the sphere bounds a 3-ball in $N$. So either $M$ or its complement is a 3-ball and in either case the proof is immediate.

For the inductive step, suppose first that $\partial M$ has multiple components $T_1, \ldots, T_n, n \geq 2$. Each component $T_i$ must bound a distinct component $J_i$ of $N - M$ since each must be separating in the non-Haken manifold $N$. Let $M' = M \cup J_n$; by inductive assumption $M'$ can be reimbedded so that its complement is standard. After the reimbedding, remove $J_n$ from $M'$, to recover a homeomorphism of $M$ and adjoin $J_1$ (now homeomorphic either to a handlebody or to $N\#(\text{handlebody})$) instead. Reimbed the resulting manifold so that its complement is standard and remove $J_1$ to recover $M$, now with standard complement.
Henceforth we can therefore assume that \( \partial M \) is connected and not a sphere. Since \( N \) is non-Haken there exists a compressing disk \( D \) for \( \partial M \).

**Case 1.** \( \partial D \) is non-separating on \( \partial M \).

If \( D \) lies inside \( M \), compress \( M \) along \( D \) to obtain \( M' \) and use the induction hypothesis to find an imbedding of \( M' \) with standard complement. Reconstruct \( M \) by attaching a trivial 1-handle to \( M' \), thus simultaneously attaching a trivial 1-handle to the complement.

If \( D \) lies outside \( M \), attach a 2-handle to \( M \) corresponding to \( D \) to obtain \( M' \), whose connected boundary has lower genus. Invoking the inductive hypothesis, imbed \( M' \) in \( N \) with standard complement. Reconstruct \( M \) from \( M' \) by removing a co-core of the attached 2-handle, thus adding a 1-handle to the complement of \( M' \).

**Case 2.** \( \partial D \) is separating on \( \partial M \).

Suppose \( D \) lies outside \( M \). Then \( D \) also separates \( J \) into two components, \( J_1 \) and \( J_2 \), since \( H_2( N ) = 0 \). Denote the components of \( \partial M - \partial D \) by \( \partial_1 \subset J_1 \) and \( \partial_2 \subset J_2 \), both of positive genus. Let \( M' = M \cup J_2 \). Reimbed \( M' \) so that its complement is standard. The boundary of \( M' \) consists of \( \partial_1 \) together with a disk. Since the complement of \( M' \) is standard, there is a non-separating compressing disk \( D' \) for \( \partial M' \) contained in the complement of \( M' \). \( D' \) is also a non-separating compressing disk for the reimbedded \( \partial M \) (which is contained in \( M' \)). Apply case 1 to this new imbedding of \( M \).

We can now suppose that the only compressing disks for \( \partial M \) are separating compressing disks lying inside \( M \). Choose a family \( \mathbf{D} \) of such \( \partial \)-reducing disks for \( M \) that is maximal in the sense that no component of \( M' = M - \mathbf{D} \) is itself \( \partial \)-compressible. Since each compressing disk is separating, \( \text{genus}(\partial M') = \text{genus}(\partial M) > 0 \) so \( \partial M' \) is compressible in \( N \). Such a compressing disk \( E \) can’t lie inside \( M' \), by construction, so it lies in the connected manifold \( N - M' \); let \( M_1 \) be the component of \( M' \) on whose boundary \( \partial E \) lies. Since each disk in \( \mathbf{D} \) was separating, \( M \) has the simple topological description that it is the boundary-connect sum of the components of \( M' \). So \( M \) can easily be reconstructed from \( M' \) in \( N - M' \) by doing boundary connect sum along arcs connecting each component of \( M' - M_1 \) to \( M_1 \) in \( N - (M' \cup E) \). After this reembedding of \( M \), \( E \) is a compressing disk for \( \partial M \) that lies outside \( M \), so we can conclude the proof via one of the previous cases. \( \square \)
4. ALIGNED FOX REIMBEDDING

Now we combine results from the previous two sections and consider
this question: If $M$ is a connected 3-submanifold of a non-Haken man-
ifold $N$ and $M$ is $\partial$-reducible, to what extent can a reimbedding of $M$,
so that its complement is standard, have its $\partial$-reducing disks aligned
with meridian disks of its complement. Obviously non-separating disks
in $M$ cannot have boundaries matched with meridian disks of $N - M$,
since $N$ contains no non-separating surfaces. But at least in the case
when $\partial M$ is connected, this is the only restriction.

**Definition 8.** For $M$ a compact irreducible orientable 3-manifold, de-
define a disjoint collection of separating $\partial$-reducing disks $D \subset M$ to be full
if each component of $M - D$ is either a solid torus or is $\partial$-irreducible.

For $M$ reducible, $D \subset M$ is full if there is a prime decomposition of $M$ so that for each summand $M'$ of $M$ containing boundary, $D \cap M'$
is full in $M'$.

$M \subset N$ a 3-submanifold is aligned to a standard complement if
the complement of $M$ is standard and there is a (complete) collection
of reducing spheres $S$ for $\partial M$ so that $S \cap M$ is a full collection of
$\partial$-reducing disks for $M$.

There is a uniqueness theorem, presumably well-known, for full col-
clections of disks, which is most easily expressed for irreducible mani-
folds:

**Lemma 9.** Suppose $M$ is an irreducible orientable 3-manifold with
boundary and $M$ is expressed as a boundary connect sum in two dif-
ferent ways: $M = M_1 \# M_2 \# \ldots \# M_n = M_1^* \# M_2^* \# \ldots \# M_n^*$, where each
$M_i, M_i^*$ is either a solid torus or $\partial$-irreducible. Then, after rearrange-
ment, $n^* = n$ and $M_i \cong M_i^*$.

**Proof.** One can easily prove the theorem from first principles, along the
lines of e. g. [H, Theorem 3.21], the standard proof of the correspond-
ing theorem for connected sum. But a cheap start is to just double $M$ along its boundary to get a manifold $DM$. The decompositions
above double to give connected sum decompositions of $DM$ in which
each factor consists of either $S^1 \times S^2$ or the double of an irreducible,
$\partial$-irreducible manifold which is then necessarily irreducible. Then [H,
Theorem 3.21] implies that $n = n^*$ and that the two original decom-
positions of $M$ also each contain the same number of solid tori. After
removing these, we are reduced to the case in which the only $\partial$-reducing
disks in $M$ are separating and $n^* = n$. 
Following the outline suggested by the proof of [H, Theorem 3.21], choose a disk $D$ that separates $M$ into the component $M_n$ and the component $M_1 \cup M_2 \cup \ldots \cup M_{n-1}$. Choose disks $E_1, \ldots, E_{n-1}$ that separate $M$ into the components $M_1', M_2', \ldots, M_n'$. Choose the disks to minimize the number of intersection components in $D \cap (\cup \{E_i\})$. Since each manifold is irreducible and $\partial$-irreducible, a standard innermost disk, outermost arc argument (in $D$) shows that in fact $D$ is then disjoint from $\{E_i\}$, so $D \subset M_n'$ (say). Since $M_n'$ is $\partial$-irreducible, $D$ is $\partial$-parallel in $M_n'$ so in fact (with no loss of generality) $M_n \cong M_n'$ and $M_1 \cup M_2 \cup \ldots \cup M_{n-1} \cong M_1' \cup M_2' \cup \ldots \cup M_n'$. The result follows by induction.

\[ \square \]

**Theorem 10.** Let $N$ be a closed orientable irreducible non-Haken 3-manifold, and $M$ be a connected compact 3-submanifold of $N$ with connected boundary. Then $M$ can be embedded in $N$ with standard complement so that $M$ is aligned to the standard complement.

**Proof.** The proof is by induction on the genus of $\partial M$. Unless $M$ has a separating $\partial$-reducing disk, there is nothing beyond the result of Theorem 7 to prove. So we assume that $M$ does have a separating $\partial$-reducing disk; in particular the genus of $\partial M$ is $g \geq 2$. We inductively assume that the theorem has been proven whenever the genus of $\partial M$ is less than $g$.

The first observation is that it suffices to find an embedding of $M$ in $N$ so that there is some reducing sphere $S$ for $\partial M$ in $N$. For such a reducing sphere divides $J = N - M$ into two components $J_1$ and $J_2$. Apply the inductive hypothesis to $M \cup J_1$ to reembed it with aligned complement $J'_1$. Notice that by a standard innermost disk argument, the reducing spheres can be taken to be disjoint from $S$. After this reembedding, apply the inductive hypothesis to $M \cup J'_1$ to reembed it so that its complement $J_2'$ is aligned. After this reembedding, $M$ has aligned complement $J'_1 \cup \partial M \backslash J'_2$.

Our goal then is to find a reembedding of $M$ so that afterwards $\partial M$ has a reducing sphere. First use Theorem 7 to reembed $M$ in $N$ so that its complement $J$ is standard, i.e. either a handlebody or $N \neq \# (\text{handlebody})$. Since $M$ is $\partial$-reducible, Lemma 3 applies: either $M$ is itself a handlebody (in which case the required reembedding of $M$ is easy) or there are disjoint compressing disks $D$ in $J$ and $E$ in $M$. Since $J$ is standard, $D$ can be chosen to be non-separating in $J$. Then $\partial E$ is not homologous to $\partial D$ in $\partial M$ so $\partial E$ is either separating in $\partial M$ or non-separating in $\partial M - \partial D$. In the latter case, two copies of $E$ can be banded together along an arc in $\partial M - \partial D$ to create a separating
essential disk in $M$ that is disjoint from $D$. The upshot is that we may as well assume that $D \subseteq J$ is non-separating and $E \subseteq M$ is separating.

Add a 2-handle to $M$ along $D$ to get $M'$, still with standard complement $J'$. Dually, $M$ can be viewed as the complement of the neighborhood of an arc $\alpha \subset M'$. If $\partial E$ is inessential in $\partial M'$, it bounds a disk $D'$ in $J' \subset J$. Then the sphere $D' \cup E$ is a reducing sphere for $M$ as required. So we may as well assume that $\partial E$ is essential in $\partial M'$ and of course still separates $M'$. By inductive assumption $M'$ can be embedded in $N$ so that its complement is aligned, but note that this does not immediately mean that $\partial E$ itself bounds a disk in $N - M'$. Let $S$ be a complete collection of reducing spheres for $\partial M'$ intersecting $M'$ in a full collection of disks.

$E$ divides $M'$ into two components, $U$ and $V$ with, say, $\alpha \subset U$. If $M'$ is reducible (i.e. contains a punctured copy of $N$) an innermost (in $E$) disk argument ensures that the reducing sphere is disjoint from $E$. By possibly tubing $E$ to that reducing sphere, we can ensure that the $N$-summand, if it lies in $M'$, lies in $U \subseteq M'$. That is, we can arrange that $V$ is irreducible. $E$ extends to a full collection of disks in $M'$, with the new disks dividing $U$ and $V$ into $\partial$-connected summands: $U = U_1 \cup \ldots \cup U_m, V = V_1 \cup \ldots \cup V_n, m, n \geq 1$, with each $U_i, V_j$ either $\partial$-irreducible or a solid torus (with one of the $U_i$ possibly containing $N$ as a connect summand). By Lemma 9, some component $V'$ of $M' - S$ is homeomorphic to $V_n$. Tube together all components of $S$ incident to $V'$ along arcs in $\partial V'$ to get a reducing sphere $S'$ dividing $M'$ into two components, one homeomorphic to $V_n$ and the other homeomorphic to $U_1V_1V_2V_3 \ldots V_{n-1}$. The latter homeomorphism carries $\alpha \subset U$ to an arc $\alpha'$ that is disjoint from the reducing sphere $S'$. Then $M' - \eta(\alpha')$ is homeomorphic to $M$ and admits the reducing sphere $S'$. In other words, the reembedding of $M$ that replaces $M' - \eta(\alpha)$ with $M' - \eta(\alpha')$ makes $\partial M$ reducible in $N$, completing the argument. \hfill \Box

**Corollary 11.** Given $M \subseteq N$ as in Theorem 10, suppose $D$ is a full set of disks in $M$. Then, with at most one exception, each component of $M - D$ embeds in $S^3$.

**Proof.** Following Theorem 10 reembed $M$ in $N$ with standard complement so that $M$ is aligned to the standard complement. Then there is a collection $S$ of disjoint spheres in $M$ so that, via Lemma 9, $M - S$ and $M - D$ are homeomorphic. Since $N$ is irreducible, each component but at most one of $N - S$ is a punctured 3-ball. Finally, each component of $N - S$ contains at most one component of $M - D$ since each component of $S$ is separating. \hfill \Box
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