SURFACES, SUBMANIFOLDS, AND ALIGNED FOX REIMBEDDING IN NON-HAKEN 3-MANIFOLDS

MARTIN SCHARLEMANN AND ABIGAIL THOMPSON

Abstract. Understanding non-Haken 3-manifolds is central to many current endeavors in 3-manifold topology. We describe some results for closed orientable surfaces in non-Haken manifolds, and extend Fox's theorem for submanifolds of the 3-sphere to submanifolds of general non-Haken manifolds. In the case where the submanifold has connected boundary, we show also that the ∂-connected sum decomposition of the submanifold can be aligned with such a structure on the submanifold's complement.

1. Introduction

A closed orientable irreducible 3-manifold N is called *Haken* if it contains a closed orientable incompressible surface; otherwise N is *non-Haken*. In Section 2 we describe some results for surfaces in non-Haken manifolds. Generalizing a theorem of Fox ([F]), we show in Section 3 that a 3-dimensional submanifold of a non-Haken manifold N is homeomorphic either to a handlebody complement in N or the complement of a handlebody in S^3. Sections 2 and 3 are independent, but both represent progress towards understanding submanifolds of non-Haken manifolds. In Section 4 we combine the techniques from Section 2 with the results from Section 3 to show that if the submanifold $M \subset N$ is ∂-reducible and has connected boundary, then the embedding can be chosen to align a full collection of separating ∂-reducing disks in M with similar disks in the complement of M.

2. Handlebodies in non-Haken manifolds

Let N be a closed orientable 3-manifold, F a closed orientable surface of non-trivial genus imbedded in N. Recall that F is *compressible* if there exists an essential simple closed curve on F which bounds an imbedded disk D in N with interior disjoint from F. D is a *compressing disk* for F.

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Definition 1. *Suppose F is a separating closed surface in an orientable irreducible closed 3-manifold N. F is* reducible *if there exists an essential simple closed curve on F which bounds compressing disks on both sides of F. The union of the two compressing disks is a* reducing sphere *for F.*

Suppose $\mathbf{S}$ is a collection of disjoint reducing spheres for F. A reducing sphere $S \in \mathbf{S}$ is redundant *if a component of $F - \mathbf{S}$ that is adjacent to $S \cap F$ is planar. $\mathbf{S}$ is* complete *if, for any disjoint reducing sphere S', S' is redundant in $\mathbf{S} \cup S'$.*

Let $\sigma(\mathbf{S})$ denote the number of components of $F - \mathbf{S}$ that are not planar surfaces.

Since N is irreducible, any sphere in N is necessarily separating. Suppose a reducing sphere S' is added to a collection $\mathbf{S}$ of disjoint reducing spheres. If S' is redundant, the number of non-planar complementary components in F is unchanged, since S' necessarily separates the component of $F - \mathbf{S}$ that it intersects and the union of two planar surfaces along a single boundary component is still planar. If S' is not redundant then the number of non-planar complementary components in F increases by one. Thus we have:

Lemma 2. *Suppose $\mathbf{S} \subset \mathbf{S}'$ are two collections of disjoint reducing spheres for F in N. Then $\sigma(\mathbf{S}) \leq \sigma(\mathbf{S}')$. Equality holds if and only if each sphere S' in $\mathbf{S}' - \mathbf{S}$ is redundant in $S' \cup \mathbf{S}$. In particular, $\mathbf{S}$ is complete if and only if for every collection $\mathbf{S}'$ such that $\mathbf{S} \subset \mathbf{S}'$, $\sigma(\mathbf{S}) = \sigma(\mathbf{S}')$.*

□

Let H be a handlebody imbedded in N. H has an *unknotted core* if there exists a pair of transverse simple closed curves $c, d \subset \partial H$ such that $c \cap d$ is a single point, d bounds an embedded disk in H and c (the *core*) bounds an imbedded disk in N.

Lemma 3. *Let F be a connected, closed, separating, orientable surface in a closed orientable irreducible 3-manifold N. Suppose that F has compressing disks to both sides. Then at least one of the following must hold:*

1. *F is a Heegaard surface for N.*
2. *N is Haken.*
3. *There exist disjoint compressing disks for F on opposite sides of F.*

Proof. The proof is an application of the generalized Heegaard decomposition described in [ST]. As F is compressible to both sides, we can construct a handle decomposition of N starting at F so that F appears as a "thick" surface in the decomposition. If F is not a Heegaard surface, then this decompostion contains a "thin" surface G adjacent to F. If G is incompressible in N, then N is Haken. If G is compressible we apply [CG] to obtain the required disjoint compressing disks for F. □

Theorem 4. *Let H be a handlebody of genus g imbedded in a closed orientable irreducible non-Haken 3-manifold N. Let G be the complement of H in N. Let $F = \partial H = \partial G$. Suppose F is compressible in G. Then at least one of the following must hold:*

(1) *The Heegaard genus of N is less than or equal to g.*
(2) *F is reducible.*
(3) *H has an unknotted core.*

Proof. The proof is by induction on the genus of H. If $g = 1$, then the result of compressing F into G is a 2-sphere, necessarily bounding a ball in N. If a ball it bounds lies in G then the Heegaard genus of N is ≤ 1. If a ball it bounds contains H then H is an unknotted solid torus in N and so it has an unknotted core.

Suppose then that $genus(H) = g > 1$ and assume inductively that the theorem is true for handlebodies of genus $g - 1$. Suppose that G, the complement of H, has compressible boundary. If G is a handlebody then $G \cup_F H$ is a Heegaard splitting of genus g and we are done. So suppose G is not a handlebody. Then by Lemma 3 there are disjoint compressing disks on opposite sides of F, say D in H and E in G. Without loss of generality we can assume that D is non-separating. Compress H along D to obtain a new handlebody H_1 with boundary F_1; let G_1 be the complement of H_1.

If ∂E is inessential in F_1 then it bounds a disk in $H_1 \subset H$ as well, so F is reducible.

If ∂E is essential in F_1 then E is a compressing disk in G_1 so we can apply the inductive hypothesis to H_1. If 1 or 3 holds then it holds for H, and we are done. Suppose instead F_1 is reducible. Let $\mathbf{S}$ be a collection of disjoint reducing spheres for F_1 chosen to maximize σ among all possible such collections and then, subject to that condition, further choose $\mathbf{S}$ to minimize $|E \cap \mathbf{S}|$. Clearly $E \cap \mathbf{S}$ contains no closed curves, else replacing a subdisk lying in the disk collection $\mathbf{S} \cap G_1$ with an innermost disk of $E - \mathbf{S}$ would reduce $|E \cap \mathbf{S}|$. Similarly, we have

Claim 1 Suppose ϵ is an arc component of $\partial E - \mathbf{S}$ and F_0 is the component of $F_1 - \mathbf{S}$ in which ϵ lies. If ϵ separates F_0 (so the ends of ϵ

necessarily lie on the same component of ∂F_0) then neither component of $F_0 - \epsilon$ is planar.

Proof of claim 1: Let c_0 be the closed curve component of $\partial F_0 \subset \mathbf{S} \cap F_1$ on which the ends of ϵ lie and, of the two arcs into which the ends of ϵ divide c_0, let α be adjacent to a planar component of $F_0 - \epsilon$. Then the curve $\epsilon \cup \alpha$ clearly bounds a disk in both G_1 and H_1 and then so does the curve $c' = \epsilon \cup (c_0 - \alpha)$. Let S' be a sphere in N intersecting F_1 in c' and S_0 be the reducing sphere in $\mathbf{S}$ containing c_0. Replacing S_0 with S' (or just deleting S_0 if c' is inessential in F_1) gives a new collection $\mathbf{S}'$ of disjoint reducing spheres, intersecting ∂E in at least two fewer points. Moreover $\sigma(\mathbf{S}') = \sigma(\mathbf{S})$ since the only change in the complementary components in F_1 is to add to one component and delete from another a planar surface along an arc in the boundary. Then the collection $\mathbf{S}'$ contradicts our initial choice for $\mathbf{S}$, a contradiction that proves the claim.

Let H' be the closed complement of $\mathbf{S}$ in H_1, so H' is itself a collection of handlebodies.

Claim 2 Either F is reducible or $\partial H'$ is compressible in $N - H'$.

Proof of claim 2: If ∂E is disjoint from $\mathbf{S}$ and is inessential in $\partial H'$, then ∂E bounds a disk in H', hence in H, so F is reducible. If ∂E is disjoint from $\mathbf{S}$ and is essential in $\partial H'$, then E compresses $\partial H'$ in $N - H'$, verifying the claim. Finally, if E intersects $\mathbf{S}$, consider an outermost disk A cut off from E by $\mathbf{S}$. According to Claim 1, this disk, together with a subdisk of $\mathbf{S}$, constitute a disk E' that compresses $\partial H'$ in $N - H'$, proving the claim.

Following Claim 2, either F is reducible or the inductive hypothesis applies to a component H_0 of H'. If 2 holds for H_0 then consider a reducing sphere S for H_0, isotoped so that the curve $c = S \cap \partial H_0$ is disjoint from the disks $\mathbf{S} \cap H_0$. The disk $S - H_0$ may intersect H_1; by general position with respect to the dual 1-handles, each component of intersection is a disk parallel to a component of $\mathbf{S} \cap H_1$. But each such disk can be replaced by the corresponding disk in $\mathbf{S} - H_1$ so that in the end c also bounds a disk in $N - H_1$. After this change, S is a reducing sphere for F_1 in N and, since c is essential in H_0, $\sigma(\mathbf{S} \cup S) > \sigma(\mathbf{S})$, contradicting our initial choice for $\mathbf{S}$. Thus in fact 1 or 3 holds for H_0, hence also for H. □

In the specific case $N = S^3$, we apply precisely the same argument, combined with Waldhausen's theorem [W] on Heegaard splittings of S^3, to obtain:

Corollary 5. *Let H be a handlebody imbedded in S^3, and suppose G, the complement of H, has compressible boundary. Then either H has an unknotted core or the boundary of H is reducible.*

This corollary is similar to ([MT], Theorem 1.1), but no reimbedding of $S^3 - H$ is required.

3. Complements of handlebodies in non-Haken manifolds

In [F] (see also [MT] for a brief version) Fox showed that any compact connected 3-dimensional submanifold M of S^3 is homeomorphic to the complement of a union of handlebodies in S^3. We generalize this result to non-Haken manifolds, showing that a submanifold M of a non-Haken manifold N has an almost equally simple description, that is, M is homeomorphic to the complement of handlebodies either in S^3 or in N.

Definition 6. *Let N be a compact irreducible* 3*-manifold, and let M be a compact* 3*-submanifold of N. We will say the complement of M in N is* standard *if it is homeomorphic to a collection of handlebodies or to $N\#$(collection of handlebodies). (We regard B^3 as a handlebody of genus* 0*.)*

Note that in the latter case M is actually homeomorphic to the complement of a collection of handlebodies in S^3.

Theorem 7. *Let N be a closed orientable irreducible non-Haken* 3*-manifold, and let M be a connected compact* 3*-submanifold of N with non-empty boundary. Then M is homeomorphic to a submanifold of N whose complement is standard.*

Proof. The proof will be by induction on $n + g$ where n is the number of components of ∂M and g is the genus of ∂M, that is, the sum of the genera of its components. If $n + g = 1$ then ∂M is a single sphere. Since N is irreducible, the sphere bounds a 3-ball in N. So either M or its complement is a 3-ball and in either case the proof is immediate.

For the inductive step, suppose first that ∂M has multiple components $T_1, \dots, T_n, n \geq 2$. Each component T_i must bound a distinct component J_i of $N - M$ since each must be separating in the non-Haken manifold N. Let $M' = M \cup J_n$; by inductive assumption M' can be reimbedded so that its complement is standard. After the reimbedding, remove J_n from M', to recover a homeomorph of M and adjoin J_1 (now homeomorphic either to a handlebody or to $N\#$(handlebody)) instead. Reimbed the resulting manifold so that its complement is standard and remove J_1 to recover M, now with standard complement.

Henceforth we can therefore assume that ∂M is connected and not a sphere. Since N is non-Haken there exists a compressing disk D for ∂M.

Case 1. ∂D is non-separating on ∂M.

If D lies inside M, compress M along D to obtain M' and use the induction hypothesis to find an imbedding of M' with standard complement. Reconstruct M by attaching a trivial 1-handle to M', thus simutaneously attaching a trivial 1-handle to the complement.

If D lies outside M, attach a 2-handle to M corresponding to D to obtain M', whose connected boundary has lower genus. Invoking the inductive hypothesis, imbed M' in N with standard complement. Reconstruct M from M' by removing a co-core of the attached 2-handle, thus adding a 1-handle to the complement of M'.

Case 2. ∂D is separating on ∂M.

Suppose D lies outside M. Then D also separates J into two components, J_1 and J_2, since $H_2(N) = 0$. Denote the components of $\partial M - \partial D$ by $\partial_1 \subset J_1$ and $\partial_2 \subset J_2$, both of positive genus. Let $M' = M \cup J_2$. Reimbed M' so that its complement is standard. The boundary of M' consists of ∂_1 together with a disk. Since the complement of M' is standard, there is a non-separating compressing disk D' for $\partial M'$ contained in the complement of M'. D' is also a non-separating compressing disk for the reimbedded ∂M (which is contained in M'). Apply case 1 to this new imbedding of M.

We can now suppose that the only compressing disks for ∂M are separating compressing disks lying inside M. Choose a family $\mathbf{D}$ of such ∂-reducing disks for M that is maximal in the sense that no component of $M' = M - \mathbf{D}$ is itself ∂-compressible. Since each compressing disk is separating, $genus(\partial M') = genus(\partial M) > 0$ so $\partial M'$ is compressible in N. Such a compressing disk E can't lie inside M', by construction, so it lies in the connected manifold $N - M'$; let M_1 be the component of M' on whose boundary ∂E lies. Since each disk in $\mathbf{D}$ was separating, M has the simple topological description that it is the boundary-connect sum of the components of M'. So M can easily be reconstructed from M' in $N - M'$ by doing boundary connect sum along arcs connecting each component of $M' - M_1$ to M_1 in $N - (M' \cup E)$. After this reimbedding of M, E is a compressing disk for ∂M that lies outside M, so we can conclude the proof via one of the previous cases. □

4. Aligned Fox reimbedding

Now we combine results from the previous two sections and consider this question: If M is a connected 3-submanifold of a non-Haken manifold N and M is ∂-reducible, to what extent can a reimbedding of M, so that its complement is standard, have its ∂-reducing disks aligned with meridian disks of its complement. Obviously non-separating disks in M cannot have boundaries matched with meridian disks of $N - M$, since N contains no non-separating surfaces. But at least in the case when ∂M is connected, this is the only restriction.

Definition 8. *For M a compact irreducible orientable 3-manifold, define a disjoint collection of separating ∂-reducing disks $\mathbf{D} \subset M$ to be* full *if each component of $M - \mathbf{D}$ is either a solid torus or is ∂-irreducible.*

For M reducible, $\mathbf{D} \subset M$ is full if there is a prime decomposition of M so that for each summand M' of M containing boundary, $\mathbf{D} \cap M'$ is full in M'.

$M \subset N$ a 3-submanifold is aligned to a standard complement *if the complement of M is standard and there is a (complete) collection of reducing spheres $\mathbf{S}$ for ∂M so that $\mathbf{S} \cap M$ is a full collection of ∂-reducing disks for M.*

There is a uniqueness theorem, presumably well-known, for full collections of disks, which is most easily expressed for irreducible manifolds:

Lemma 9. *Suppose M is an irreducible orientable 3-manifold with boundary and M is expressed as a boundary connect sum in two different ways: $M = M_1 \natural M_2 \natural \dots \natural M_n = M_1^* \natural M_2^* \natural \dots \natural M_{n^*}^*$, where each M_i, M_j^* is either a solid torus or ∂-irreducible. Then, after rearrangement, $n^* = n$ and $M_i \cong M_i^*$.*

Proof. One can easily prove the theorem from first principles, along the lines of e. g. [H, Theorem 3.21], the standard proof of the corresponding theorem for connected sum. But a cheap start is to just double M along its boundary to get a manifold DM. The decompositions above double to give connected sum decompositions of DM in which each factor consists of either $S^1 \times S^2$ or the double of an irreducible, ∂-irreducible manifold which is then necessarily irreducible. Then [H, Theorem 3.21] implies that $n = n^*$ and that the two original decompositions of M also each contain the same number of solid tori. After removing these, we are reduced to the case in which the only ∂-reducing disks in M are separating and $n^* = n$.

Following the outline suggested by the proof of [H, Theorem 3.21], choose a disk D that separates M into the component M_n and the component $M_1 \natural M_2 \natural \dots \natural M_{n-1}$. Choose disks $E_1, \dots, E_{n-1}$ that separate M into the components $M_1^*, M_2^*, \dots M_{n^*}^*$. Choose the disks to minimize the number of intersection components in $D \cap (\cup\{E_i\})$. Since each manifold is irreducible and ∂-irreducible, a standard innermost disk, outermost arc argument (in D) shows that in fact D is then disjoint from $\{E_i\}$, so $D \subset M_n^*$ (say). Since M_n^* is ∂-irreducible, D is ∂-parallel in M_n^* so in fact (with no loss of generality) $M_n \cong M_n^*$ and $M_1 \natural M_2 \natural \dots \natural M_{n-1} \cong M_1^* \natural M_2^* \natural \dots \natural M_{n-1}^*$. The result follows by induction. □

Theorem 10. *Let N be a closed orientable irreducible non-Haken 3-manifold, and M be a connected compact 3-submanifold of N with connected boundary. Then M can be reimbedded in N with standard complement so that M is aligned to the standard complement.*

Proof. The proof is by induction on the genus of ∂M. Unless M has a separating ∂-reducing disk, there is nothing beyond the result of Theorem 7 to prove. So we assume that M does have a separating ∂-reducing disk; in particular the genus of ∂M is $g \geq 2$. We inductively assume that the theorem has been proven whenever the genus of ∂M is less than g.

The first observation is that it suffices to find an embedding of M in N so that there is some reducing sphere S for ∂M in N. For such a reducing sphere divides $J = N - M$ into two components J_1 and J_2. Apply the inductive hypothesis to $M \cup J_1$ to reimbed it with aligned complement J_2'. Notice that by a standard innermost disk argument, the reducing spheres can be taken to be disjoint from S. After this reimbedding, apply the inductive hypothesis to $M \cup J_2'$ to reimbed it so that its complement J_1' is aligned. After this reimbedding, M has aligned complement $J_1' \cup_{S-M} J_2'$.

Our goal then is to find a reimbedding of M so that afterwards ∂M has a reducing sphere. First use Theorem 7 to reimbed M in N so that its complement J is standard, i. e. either a handlebody or $N\#$ (handlebody). Since M is ∂-reducible, Lemma 3 applies: either M is itself a handlebody (in which case the required reimbedding of M is easy) or there are disjoint compressing disks D in J and E in M. Since J is standard, D can be chosen to be non-separating in J. Then ∂E is not homologous to ∂D in ∂M so ∂E is either separating in ∂M or non-separating in $\partial M - \partial D$. In the latter case, two copies of E can be banded together along an arc in $\partial M - \partial D$ to create a separating essential disk in M that is disjoint from D. The upshot is that we may as well assume that $D \subset J$ is non-separaring and $E \subset M$ is separating.

Add a 2-handle to M along D to get M', still with standard complement J'. Dually, M can be viewed as the complement of the neighborhood of an arc $\alpha \subset M'$. If ∂E is inessential in $\partial M'$, it bounds a disk D' in $J' \subset J$. Then the sphere $D' \cup E$ is a reducing sphere for M as required. So we may as well assume that ∂E is essential in $\partial M'$ and of course still separates M'. By inductive assumption M' can be embedded in N so that its complement is aligned, but note that this does not immediately mean that ∂E itself bounds a disk in $N - M'$. Let $\mathbf{S}$ be a complete collection of reducing spheres for $\partial M'$ intersecting M' in a full collection of disks.

E divides M' into two components, U and V with, say, $\alpha \subset U$. If M' is reducible (i.e. contains a punctured copy of N) an innermost (in E) disk argument ensures that the reducing sphere is disjoint from E. By possibly tubing E to that reducing sphere, we can ensure that the N-summand, if it lies in M', lies in $U \subset M'$. That is, we can arrange that V is irreducible. E extends to a full collection of disks in M', with the new disks dividing U and V into ∂-connected sums: $U = U_1 \natural \ldots \natural U_m, V = V_1 \natural \ldots \natural V_n, m, n \geq 1$, with each U_i, V_j either ∂-irreducible or a solid torus (with one of the U_i possibly containing N as a connect summand). By Lemma 9, some component V' of $M' - \mathbf{S}$ is homeomorphic to V_n. Tube together all components of $\mathbf{S}$ incident to V' along arcs in $\partial V'$ to get a reducing sphere S' dividing M' into two components, one homeomorphic to V_n and the other homeomorphic to $U \natural V_1 \natural V_2 \natural \ldots \natural V_{n-1}$. The latter homeomorphism carries $\alpha \subset U$ to an arc α' that is disjoint from the reducing sphere S'. Then $M' - \eta(\alpha')$ is homeomorphic to M and admits the reducing sphere S'. In other words, the reimbedding of M that replaces $M' - \eta(\alpha)$ with $M' - \eta(\alpha')$ makes ∂M reducible in N, completing the argument. □

Corollary 11. *Given $M \subset N$ as in Theorem 10, suppose $\mathbf{D}$ is a full set of disks in M. Then, with at most one exception, each component of $M - \mathbf{D}$ embeds in S^3.*

Proof. Following Theorem 10 reimbed M in N with standard complement so that M is aligned to the standard complement. Then there is a collection $\mathbf{S}$ of disjoint spheres in M so that, via Lemma 9, $M - \mathbf{S}$ and $M - \mathbf{D}$ are homeomorphic. Since N is irreducible, each component but at most one of $N - \mathbf{S}$ is a punctured 3-ball. Finally, each component of $N - \mathbf{S}$ contains at most one component of $M - \mathbf{D}$ since each component of $\mathbf{S}$ is separating. □

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Mathematics Department, University of California, Santa Barbara, CA 93106, USA

E-mail address: mgscharl@math.ucsb.edu

Mathematics Department, University of California, Davis, CA 95616, USA

E-mail address: thompson@math.ucdavis.edu