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TUNNEL NUMBER ONE KNOTS SATISFY THE POENARU CONJECTURE

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It is shown that tunnel number one knots satisfy the Poenaru conjecture and so have Property *R*. As a sidelight they are also shown to be doubly prime.

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Introduction

The *tunnel number* of a *PL* knot K is the minimum number of *PL* one-cells which must be attached in order that the regular neighborhood of the resulting complex has complement a handlebody [2]. It is easy to see that n-bridge knots have tunnel number $\leqslant (n-1)$ and torus knots have tunnel number one. More difficult is finding knots which have higher tunnel number. If a knot has tunnel number one, it will be a one-relator knot and therefore prime [6] (for a geometric proof that tunnel number one knots are prime see 2.2). Among prime knots, those with nonvanishing second elementary ideal are not one-relator and therefore not tunnel number one. It seems a good but difficult conjecture that one-relator knots coincide with tunnel number one knots.

The goal here is to show that tunnel number one knots satisfy two other properties: they are doubly prime (that is, they cannot be written as the join of two prime tangles) and they satisfy the Poenaru conjecture (that is, no $2k+1$ longitudes of the knot bound an incompressible, boundary incompressible planar surface in the complement of a tubular neighborhood of the knot). The first property is really a curiosity—it is easily proven using the techniques developed elsewhere in the paper and so is included here. The second property is the crucial difficult step in the solution of the genus two Schoenflies conjecture [8].

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The outline of the paper is as follows: In Section 1 there is an elaborate lemma, apparently a broad generalization of a theorem of Tsukui [9]. In Section 2 there is the proof that tunnel number one knots are doubly prime, together with a geometric proof that they are prime. In Section 3 we begin work on the Poenaru conjecture; we suppose a tunnel number one knot has an odd number of longitudes bounding an incompressible planar surface P in the knot complement. First we show that the tunnel can be pushed off of P, and then minimize the complexity of the intersection of P with a certain compressing disk E in the complement of the union of knot and tunnel. This gives rise to a combinatorial object called a bilabelled tree which is studied in depth in Section 4. In Section 5 the proof is completed. The combinatorics involved is rather intimidating; it would be interesting to have other applications (see also [7]).

1. Pushing tunnels off punctured spheres

Let γ be a PL simple closed curve in an orientable 3-manifold M, with tubular neighborhood $\gamma \times D^2$. We say that γ is a *k-string composite* if there is a PL 2-sphere S^2 in M intersecting $\gamma \times D^2$ in k meridional disks, such that $S^2-(\gamma \times \mathring{D}^2)$ is incompressible and ∂-incompressible in $M-(\gamma \times \mathring{D}^2)$. For example (see Section 2) a knot in S^3 is prime if and only if it is not a 2-string composite and a prime knot is doubly prime if and only if it is not a 4-string composite. Note that if a knot is a $(2k+1)$-string composite, the 2-sphere is non-separating.

Suppose γ is a k-string composite in M and $(I, \partial I) \subset (M-(\gamma \times \mathring{D}^2), \gamma \times \partial D^2)$ is a PL arc, called, occasionally, the tunnel. Denote by W the complement in $M-(\gamma \times \mathring{D}^2)$ of a relative regular neighborhood of I.

1.1 Lemma. *If M contains no Lens space summands and W contains a boundary non-separating, boundary reducing disk, then there is a 2-sphere disjoint from I which decomposes γ into an l-string composite $0<l\leq k$. Moreover, if k is odd, so is l.*

Remarks. Tsukui [Ts, 3.6] treated the special case in which M is $S^1 \times S^2$, γ is $S^1 \times (\text{pt.})$ and the sphere is $(\text{pt.}) \times S^2$.

The requirement that M contains no Lens space summands may be unnecessary, but does greatly simplify the problem.

Proof of 1.1. We can assume that γ is an l-string composite $0<l<k$ if and only if k is odd and l is even. Choose a 2-sphere S' which k-string decomposes γ, which intersects I as well as γ transversally and which, among all such, has fewest number of intersections with I. Among all such S' and all compressing disks E for ∂W in W such that ∂E does not separate ∂W, and E is transverse to S', choose those for which the number of components of $S' \cap E$ is minimal.

Denote that portion of S' which lies in $M-(\gamma\times\mathring{D}^2)$ by S, a planar surface with boundary components $\{b_i\}$ $(1\leq i\leq k)$ meridians in $\gamma\times\partial D^2$. A standard innermost circle argument shows that $E\cap S$ consists of a finite collection of arcs in W.

Let $n=\#(I\cap S)$. It will be convenient to reparametrize so that $I=[0,n+1]$ and the points of $I\cap S$ are integers $1\leq i\leq n$. We can assume that the complement of W in S consists of n disjoint disks N_i, each N_i containing $i\in I$. Let N_{ij} denote the annulus in ∂W lying between N_i and N_j, for $1\leq i\neq j\leq n$.

Consider $S\cap E$, a collection of arcs in E; each end of each arc is a point in some ∂N_i or b_i; assign the label i to the point if it lies in ∂N_i and label it b_i if it lies in that component of ∂S.

As we read the labels around ∂E, it follows from the continuity of the attaching map of ∂E and the minimization of $\#(\partial E\cap S)$ that the sequence of labels can be read off the following 'train track'.

(In the figure we assume that the b_i are labelled in order around γ and the ends of I are attached in $\gamma\times\partial D^2$ to the components of $(\gamma\times\partial D^2)-S$ lying between b_k and b_1 and between b_r and b_{r+1}. No change is needed in the argument if in fact both ends lie between b_k and b_1.)

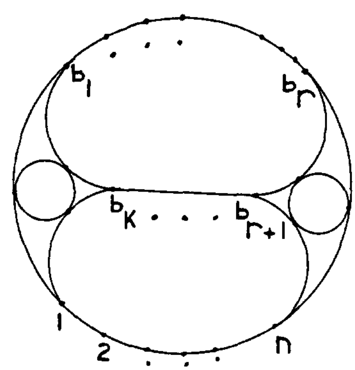


Fig. 1.

Notice that if $n>1$, then contained in any arc of $\partial E-S$ bounded by labels n, n or 1, 1 is a loop in $\gamma\times\partial D^2$ which begins and ends at a single point of I.

Claim 1. *Such a loop in $\gamma\times\partial D^2$ must be a meridian.*

Proof of Claim 1. The only alternative, since the loop is disjoint from the meridians of ∂S, is that the loop bounds a disk in $\gamma\times\partial D^2$. If this disk (based at 0, for example) does not contain the end $n+1$ of I, then pull ∂E across the disk and up into I, reducing $\#(\partial E\cap S)$. If the disk does contain $n+1$ the same argument shows that every time ∂E leaves I through $n+1$ it must immediately reenter I through 0 (see Fig. 2). Then ∂E intersects one component of $\partial N_{1,n}$ more often than the other, so again ∂E can be isotoped to intersect S less often. $\square$

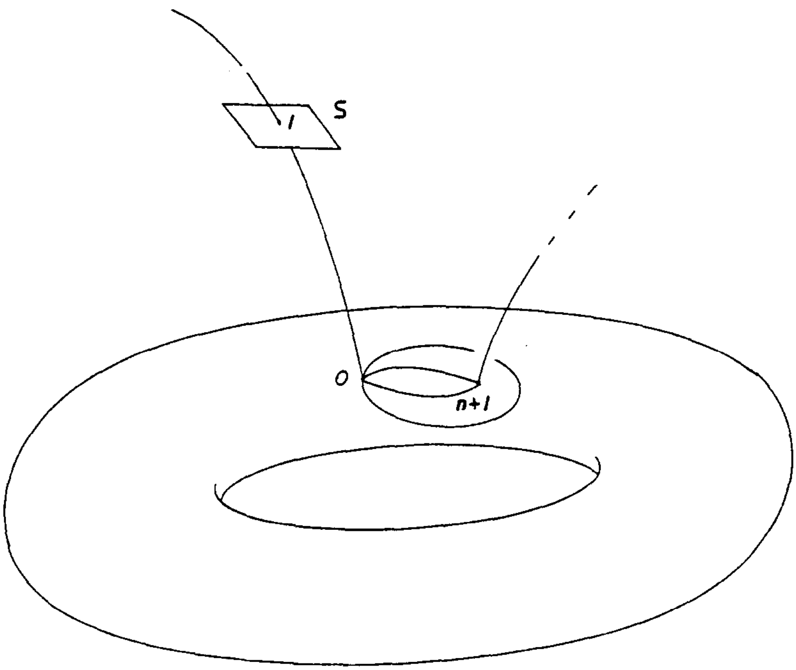


Fig. 2.

There is a dual tree T imbedded in E with vertices a point from each 2-cell of $E-S$; edges connect vertices of adjacent 2-cells. Each edge corresponds to an arc in $E \cap S$, and we label it on both sides with the labels of the corresponding arc in $E \cap S$. For an arbitrary connected tree T imbedded in the plane we use the following definitions:

An *outermost vertex* is a vertex which is in the boundary of a single edge, which is called an *outermost edge*. A *fork* is a vertex in the boundary of three or more edges. Let F be the collection of forks of T and remove from T all components of $T-F$ which contain an outermost vertex. An outermost vertex of the resultant tree is an *outermost fork*. (If T has no forks then let any vertex *not* an outermost vertex be called an outermost fork.) If v is an outermost fork then the components of $T-\{v\}$ which contain no forks are called *outermost lines* of v. If v is any fork then two components of $T-v$ are *adjacent* if a small circle around v in the plane contains an arc intersecting only those components of $T-v$.

Now examine the labelled tree $T \subset E$ arising from $E \cap S$ as described above.

Claim 2. *Any outermost edge is labelled* $\{1, 1\}$, $\{n, n\}$, *or* $\{1, n\}$.

Proof of Claim 2. From Fig. 1, the alternatives are

a) $\{i, i+1\}$, $1 \leq i \leq n-1$,
b) $\{b_i, b_j\}$, $i \neq j$,

c) $\{b_i, b_i\}$,

d) $\{1, b_i\}, \{n, b_i\}$.

In cases a) and d) one could use the outermost 2-cell C of $E - S$ to reduce $\#(I \cap S)$.

In cases *b*) and *c*) we can assume that the arc $\partial C \cap \partial E$ does not run over I, for if it did then I would be disjoint from S and we would be through. Since S is boundary incompressible, $\alpha = \partial C \cap S$ must be an arc isotopic in S to an arc β in ∂S. This eliminates possibility b); in case c) replace the cell in S whose boundary is $\alpha \cup \beta$ by C. This replaces S by a genus zero surface with the same number of boundary components, not increasing $\#(I \cap S)$ and decreasing the number of components of $E \cap S$. □

Claim 3. *If $n \geq 2$, no outermost edge is labelled* $\{1, 1\}$ *or* $\{n, n\}$.

Proof of Claim 3. The cases are symmetric, so consider only $\{1, 1\}$.

The union of regular neighborhoods of the outermost 2-cell in $E - S$ and of the interval $[0, 1]$ in I is a bicollared cylinder in $M - (\gamma \times \mathring{D}^2)$, intersecting $\gamma \times \partial D^2$ in one of its ends and intersecting S only in its other end. Attach a regular neighborhood of S to this bi-collared cylinder. The resulting 3-manifold has three boundary components, one parallel to S.

By Claim 1, the end of the cylinder in $\gamma \times \partial D^2$ is a meridian, so the union of the other two components has boundary consisting of $k+2$ meridians, at least one in each component, and intersects I in at least one fewer point.

The end of the cylinder in S must be an essential loop in S, for otherwise the cylinder together with the disk it bounds in S would be compressing disk for ∂W which has fewer (indeed no) intersections with S.

One component (call it Q) must have k or fewer boundary components, and, if k is odd, we may choose Q to have an odd number of boundary components. Clearly Q is incompressible in $M - (\gamma \times \mathring{D}^2)$ because it is homotopic therein to a sub-planar surface of S with essential boundary components.

Since Q has fewer boundary components than S, it would improve on S unless Q is boundary compressible in $M - (\gamma \times D^2)$. Consider, then, a cell C with $\partial C = \alpha \cup \beta$, α in $\gamma \times \partial D^2$ and β a non boundary parallel arc in Q. If both points of $\partial \alpha$ lie in the same component of ∂Q then α together with a subarc λ of ∂Q must bound a disk in $\gamma \times \partial D^2$, hence, by incompressibility of Q, $\beta \cup \lambda$ also bounds a disk in Q. This cannot happen, since β is not ∂-parallel. If both points of $\partial \beta$ lie in different components of ∂S, then C would contradict the ∂-incompressibility of S in $M - (\gamma \times \mathring{D}^2)$. Thus one end of β lies in a component of ∂S and the other in the component of ∂Q not in ∂S, namely the end of the cylinder which lies in $\gamma \times \partial D^2$. In fact we can assume that, in that cylinder, β coincides with the subinterval $[0, 1]$ of I.

Now return to the original S. The cell C has boundary consisting of three pieces: an arc α in $\gamma \times \partial D^2$, an arc in S, and $[0, 1] \subset I$. From the ∂-incompressibility of S we can take $S \cap \mathring{C} = \emptyset$ and therefore use C to isotope S so that $\#(I \cap S)$ is reduced. □

Claim 4. *For $2 \leqslant i \leqslant n-1$, the pattern*

$$\begin{array}{cc} i & i+1 \\ \hline i & i-1 \end{array}$$

never occurs in T.

Proof of Claim 4. Consider the corresponding arc of $S \cap E$ with ends labelled $\{i, i\}$ (Fig. 3). The dotted arc shown lies in a small neighborhood of S in $M-(\gamma \times \mathring{D}^2)$ and its ends can be connected by a small arc in the interior of $N_{i-1,i}$. The result is a circle in a neighborhood of S in W, intersecting S in exactly one point. Since S is 2-sided, this is impossible. $\square$

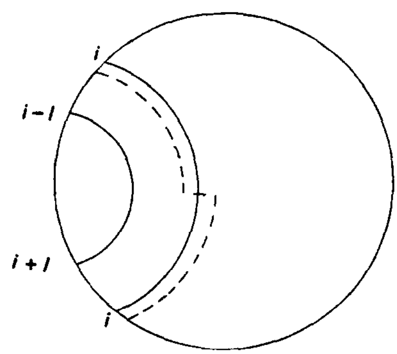
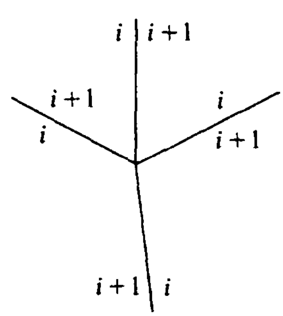


Fig. 3.

Claim 5. *No outermost edge of T is labelled* $\{1, 1\}$.

Proof of Claim 5. This is Claim 3 if $n \geqslant 2$. The same proof works for $n=1$ if the arc of ∂E in the outermost 2-cell of $E-S$ does not run down one end of I and up the other. But the arc cannot do this or, as in Claim 4, one could show that S is one-sided in $M-(\gamma \times D^2)$. $\square$

Claim 6. *If $n \geqslant 3$, then for no $1 \leqslant i \leqslant n-1$ is there a vertex in T all of whose p adjacent edges, $p \geqslant 2$, are labelled as shown*



If all but one of the p edges is outermost, the same is true for $n=2$.

Proof of Claim 6. The corresponding 2-cell F of $E-S$ has boundary (oriented clockwise in Fig. 4) consisting of arcs running from N_i to N_{i+1} in S and from N_{i+1} to N_i in $N_{i,i+1}$. (Use the proof of Claim 2 to verify this in case $n=2$.) Let N denote a regular neighborhood in M of the 2-sphere S'. Then ∂F intersects the meridian N_i of the punctured solid torus $N \cup ([i, i+1] \times D^2)$ algebraically p times and bounds the disk F in its complement.

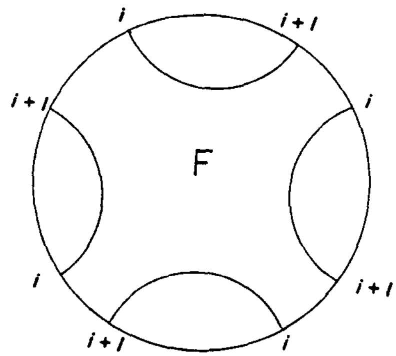


Fig. 4.

Then $F \cup N \cup N_{i,i+1}$ has regular neighborhood in M a double punctured Lens space with $|H_i(L)| = p$. This contradicts the hypothesis on M. $\square$

Claim 7. *The number of edges in two adjacent outermost lines of T must total at least n.*

Proof of Claim 7. By Claims 2 and 3 all outermost edges are labelled $\{1, n\}$. From Fig. 1 it follows that the arc in ∂E corresponding to the path in T from one outermost vertext to another must pass through at least n labels. $\square$

Claim 8. *No outermost line has more than $n/2$ edges.*

Proof of Claim 8. If it does, then, since the outermost vertex is labelled $\{1, n\}$, the outermost line contains one of the following patterns, depending on whether n is odd or even.

$$\cdots \frac{p}{p} \bullet \frac{p+1}{p-1} \cdots \frac{2p-2}{2} \bullet \frac{2p-1}{1} \qquad n = 2p-1$$

$$\cdots \frac{p}{p+1} \bullet \frac{p+1}{p} \cdots \frac{2p-1}{2} \bullet \frac{2p}{1} \qquad n = 2p.$$

One case contradicts Claim 4, the other Claim 6. $\square$

Claim 9. *T has no outermost forks.*

Proof of Claim 9. Combining Claims 7 and 8, each outermost line of an outermost fork must have length $n/2 = p$. Hence the labelling around the fork is as in Fig. 5.

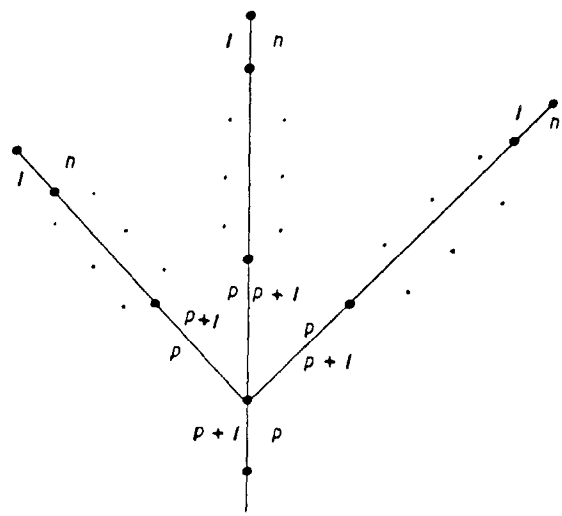


Fig. 5.

This contradicts Claim 6. □

Proof of 1.1. By Claim 9, T has at most a single edge. If T is an edge then, by Claim 7, $n = 1$ and T is labelled $\{1, 1\}$. This contradicts Claim 5. Hence T is a point and so $S \cap E = \emptyset$. Then either $n = 0$ and we are done, or $\partial E \subset W \cap (\gamma \times \partial D^2)$. In the latter case, either ∂E is a meridian of $\gamma \times \partial D^2$, so we take E for S (and γ is a 1-string composite), or else E bounds a disk F in $\gamma \times \partial D^2$, but no disk in ∂W. In this case, since ∂E is non-separating, the disk F must contain a single end of I. Then ∂E is a meridian of I, so E can be used to pipe off any intersections of S with I. □

2. Doubly prime knots

A *tangle* (B, t) is a 3-ball B containing a pair of disjoint proper PL imbedded arcs t, called strings; a tangle is *prime* if

1) any 2-sphere imbedded in B which meets t transversally in two points bounds a 3-ball in B which meets t is a single unknotted arc.

2) no properly imbedded disk separates the strings.

A knot $\gamma \subset S^3$ is *doubly prime* if there is no 2-sphere which intersects γ transversally in S^3 and divides γ into two prime tangles. In the terms of Section 1:

Lemma 2.1. *A prime knot $\gamma \subset S^3$ is doubly prime if and only if it is not a 4-string composite.*

Proof. Suppose γ is a prime knot which is a 4-string composite; then there is a *PL* 2-sphere S' in S^3, dividing γ into two tangles (A, γ_A) and (B, γ_B). Let W_A (W_B) denote the closure of the complement of a regular neighborhood of γ_A in A (γ_B in B). Then $\partial A \cap W_A$ and $\partial B \cap W_B$ are both incompressible, so 2) is satisfied.

If a 2-ball B' in B, say, intersects γ_B in a single *knotted* arc then, since γ is prime, if we replace the knotted arc by an unknotted arc, γ becomes the unknot. But by Van Kampen's theorem and the disk theorem, if the unknot is decomposed into two tangles, one or the other contains a disk separating the strings. This property is clearly preserved if we replace an unknotted segment of a string by a knotted one, contradicting the incompressibility of $\partial A \cap W_A$ and $\partial B \cap W_B$. This contradiction shows that both tangles are prime.

On the other hand, suppose γ is not doubly prime, so that γ is decomposed into two prime tangles which we continue to denote by (A, γ_A) and (B, γ_B) etc., as above. By property 2) and the disk theorem $\partial A \cap W_A = \partial B \cap W_B$ is incompressible. Suppose that $\partial A \cap W_A = \partial B \cap W_B$ is ∂-compressible in $W_A \cup W_B$, the complement of a regular neighborhood of γ in S^3. Indeed, let C be a cell in e.g. A with $\partial C = \alpha \cup \beta$, α an arc in $\partial A \cap W_A$ and β an arc in $\partial W_A - \mathrm{int}(\partial A \cap W_A)$, such that β is not isotopic rel end points to a curve in ∂A. Since $\partial W_A - \mathrm{int}(\partial A \cap W_A)$ consists of two cylinders, parallel to the strings of γ_A, β must run from one end of a cylinder to the other. Then the two sides of a regular neighborhood of C, together with the complement of the neighborhood of the cylinder is a 2-disk in A separating the 2-strings. The contradiction shows $\partial A \cap W_A = \partial B \cap W_B$ is ∂-incompressible and so γ is a 4-string composite. □

Before showing that tunnel number one knots are doubly prime, as a warm-up we show

Theorem 2.2. (Norwood.) *Tunnel number one knots are prime.*

Proof. Suppose γ were a tunnel number one knot and a 2-string composite (i.e. not a prime knot). Apply 1.1 with I the tunnel, and E a non-separating, ∂-reducing disk in the handlebody complement W of a regular neighborhood of $\gamma \cup I$. From 1.1 we conclude that there is a connected sum decomposition $\gamma = \alpha \# \beta$, where α and β are non-trivial knots, such that the tunnel is disjoint from the 2-sphere on which the connected sum takes place. Suppose the tunnel lies on the β side of this 2-sphere. Then $\pi_1(S^3 - \alpha) \underset{Z}{*} \pi_1(S^3 - (\beta \cup \mathrm{tunnel})) \simeq \pi_1(W) \simeq Z * Z$. Hence $\pi_1(S^3 - \alpha)$ is a subgroup of a free group, hence free. But then α is unknotted, a contradiction. □

Theorem 2.3. *Tunnel number one knots are doubly prime.*

Proof. Suppose γ is a tunnel number one knot in S^3 which is a 4-string composite. For homological reasons γ is not a 1-string or 3-string composite, and 2.1 shows that γ is not a 2-string composite. By 1.1 and 2.1 there is a 2-sphere in S^3 separating (S^3, γ) into two prime tangles, (A, γ_A) and (B, γ_B) so that the tunnel I lies entirely in, say, B. Let W denote the closure of the complement of a regular neighborhood of $\gamma \cup I$; that part of the 2-sphere in W is a planar surface S with four boundary components ρ_i, $1 \leq i \leq 4$. Denote $W \cap A$ by W_A and $W \cap B$ by W_B. Since $\pi_1(S) \to \pi_1(A - \gamma_A)$ and $\pi_1(S) \to \pi_1(B - \gamma_B)$ are injective, so are $\pi_1(S) \to \pi_1(W_A)$ and $\pi_1(S) \to \pi_1(W_B)$ (all homomorphisms induced by inclusion). Thus, letting F_i denote the free group on i generators,

$$F_2 \simeq \pi_1(W) \simeq \pi_1(W_A) \underset{\pi_1(S)}{*} \pi_1(W_B).$$

Since $\pi_1(S) \simeq F_3$ it follows that $\pi_1(W_A) \simeq F_2$ and $\pi_1(W_B) \simeq F_3$.

Claim 1. *$i_* : \pi_1(S) \to \pi_1(W_B)$ is not surjective.*

Proof of Claim 1. The inclusion induced $j_* : \pi_1(W_B) \to \pi_1(B - \gamma_B)$ is surjective by construction. Since (B, γ_B) is prime, $j_* i_* : \pi_1(S) \to \pi_1(B - \gamma_B)$ is injective. On the other hand j_* is not an isomorphism, since $H_1(W_B) \simeq Z \oplus Z \oplus Z$ and $H_1(B - \gamma_B) \simeq Z \oplus Z$. Thus j_* is not injective, so i_* cannot be surjective. $\square$

Now proceed in a manner which is formally similar to the proof of 1.1. Let E be a non-separating compressing disk for ∂W in the handlebody W, isotoped to be transverse to S, to intersect S only in arc components, and chosen to minimize the number of components of $S \cap E$. Choose the labels ρ_i so that ρ_1 and ρ_2 (hence ρ_3 and ρ_4) are in the same cylinder components of $\partial W \cap A$. Schematically, γ and the ρ_i appear in each tangle as shown in Fig. 6.

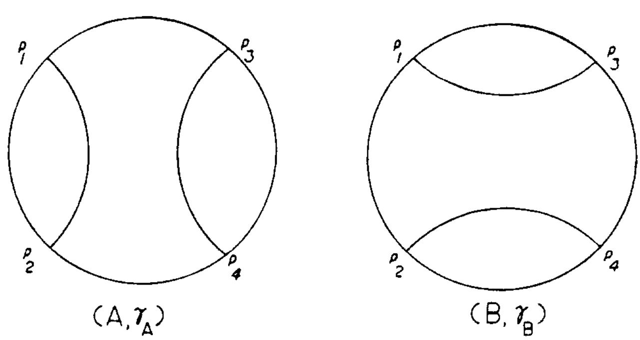


Fig. 6.

Now consider the tree $T \subset E$ coming from the intersection of S with E. Once again the end of each arc of $S \cap E$ corresponds to a point lying in one of the ρ_i, $1 \leq i \leq 4$, and we label the corresponding side of an edge of T by i.

Remark. The vertices of T alternate between those that correspond to cells in A and those that correspond to cells in B.

Claim 2. *Any outermost vertex in T corresponds to a cell in B.*

Proof of Claim 2. The boundary of a cell corresponding to an outermost vertex is the union of an arc in S and an arc in ∂W. Now $W \cap A$ is just a regular neighborhood of γ_A, so if the cell were in A, (A, γ_A) would not be prime. □

Claim 3. *Any outermost line of T has length 1.*

Proof of Claim 3. Suppose an outermost line of T had length $\geqslant 2$. Then adjacent to an outermost vertex would be a vertex corresponding to a cell C in A whose boundary consists of two arcs in S and two arcs in ∂W.

If the arcs of $\partial C \cap \partial W$ run parallel to distinct strings of γ_A, consider the complement W'_A of a relative regular neighborhood of C in W_A. Now W_A is obtained from W'_A by attaching a 1-handle dual to C. Thus $F_2 \simeq \pi_1(W_A) \simeq Z * (W'_A)$, so $\pi_1(W'_A) \simeq Z$. It follows that W'_A is a solid torus. In particular, there is a disk in W'_A whose boundary is the union of an arc in $\partial W'_A$ parallel to a string of γ_A and an arc in S. This again contradicts primality of (A, γ_A).

If the arcs of $\partial C \cap \partial W$ run parallel to a single string of γ_A, the complement of a regular neighborhood of C in W_A has one component a ball containing the other string. Since (A, γ_A) is prime, it follows from property 1) of prime tangles that both strings are, in fact, parallel, and the argument reduces to the previous case. □

Call a disk C in W_B a compressing disk of type (i, j), $1 \leqslant i, j \leqslant 4$, if $\partial C = \alpha \cup \beta$, α a boundary non-parallel arc in S, β an arc in $\partial W_B - S$, such that one point of $\alpha \cap \beta$ lies in ρ_i and the other in ρ_j.

Claim 4. *Suppose both ends of the tunnel are connected to the same component of γ_B and C_1 and C_2 are disjoint compressing disks of type (i, j) and (k, l) respectively. Then $i \equiv j \equiv k \equiv l \bmod 2$.*

Proof of Claim 4. The circles ρ_i and ρ_j are connected in $\partial W_B - S$ if and only if $i \equiv j \bmod 2$. Furthermore no compressing disk can involve the component of γ_B to which the tunnel is not attached, since (B, γ_B) is prime. Thus $i \equiv j \equiv k \equiv l \bmod 2$.

Claim 5. *Suppose the ends of the tunnel are connected to distinct components of γ_B and C_1 and C_2 are disjoint compressing disks of type (i, j) and (k, l) respectively. Then either*

a) *three of i, j, k, l are equal or*

b) $\{i, j\} = \{k, l\}$.

Furthermore, in case a), *if* $i=j=k\neq l$ *then* ρ_l *and* ρ_m, $m\neq i, l$ *lie in different components of* $S-C_1$, *and any arc of* $S-C_1$ *with both ends in* ρ_l *is isotopic rel end points in* $S\cap W$ *to a subarc of* ρ_l.

Proof of Claim 5. Suppose i, j, k, l are all distinct; with no loss of generality take $(i,j)=(1,3)$ and $(k,l)=(2,4)$.

Let W'_B denote the complement of a regular neighborhood of $C_1\cup C_2$ in W_B. Then W_B is obtained from W'_B by attaching two 1-handles dual to C_1 and C_2. Thus

$$F_3=\pi_1(W_B)=\pi_1(W'_B)*Z*Z$$

so $\pi_1(W'_B)\simeq Z$. Note that W'_B is then just a solid torus, with $\pi_1(W'_B)$ generated by a loop λ_0 in $S-(C_1\cup C_2)$. Since C_1 and C_2 intersect S in single arcs, there are loops λ_1 and λ_2 in S based at a point in λ_0, which intersect C_1 and C_2 respectively in a single point. Then the λ_i freely generate $\pi_1(W_B)$, contradicting Claim 1.

Essentially the same argument applies in case $i\neq j$ and $k\neq l$ and only three of $\{i,j,k,l\}$ are distinct.

Suppose then that $i=j=1$, say. Then $\partial C\cap S$ is an arc which divides S into two components and $\partial C-S$ is an arc which divides the planar ∂W_B-S into two components. Hence ∂C separates ∂W_B and, since $H_1(\partial W_B)\to H_1(W_B)$ is surjective, it follows that C separates W_B. Two complete ρ_i's lie in one component of $\partial W_B-\partial C$ and one, say ρ_3, lies in the other. Remove a regular neighborhood of C from W_B, obtaining W'_B. Then W_B is obtained from the two components of W'_B by attaching the 1-handle dual to C in its regular neighborhood. Hence $\pi_1(W_B)$ ($\simeq F_3$) is the free product of the fundamental groups of the two components of W'_B, so the component of W'_B containing ρ_3 is a solid torus, with ρ_3 generating its fundamental group. A meridian for this torus which intersects ρ_3 once and is disjoint from the boundary of the regular neighborhood of C will be a compressing disk of type $(1,3)$. Unless $\{k,l\}=\{1,3\}$ (situation a) of the hypothesis) or $k=l$ the pair $(1,3)$ and (k,l) contradicts the previous two cases. If $k=l$ then, since $C_1\cap C_2=\emptyset$, either $k=l=i=j$ (situation b) of the hypothesis) or $k=l=2$ or 4). Repeating the argument above, there is also a disjoint compressing disk of type $(2,4)$. The pair $(1,3)$ and $(2,4)$ contradicts the first case. To verify the last part of claim 5 (for $i=j=k=1$, $l=3$) notice that ρ_2 and ρ_4 are separated in $S\cap W$ from ρ_3 by the arc $\partial C_1\cap S$ and that the component of $(S\cap W)-\partial C_1$ containing ρ_3 is an annulus with ρ_3 one of its boundary components. □

Claim 6. *The tree T has no outermost forks.*

Proof of Claim 6. By Claim 3 all outermost lines are also outermost edges. By Claim 2 and the remark preceding it the cell C corresponding to the outermost fork lies in W_A. In particular, each component of $\partial C\cap\partial W$ is an arc running parallel to a string of γ_A. Thus adjacent labels of adjacent outermost edges at the fork must either be the pair $\{1,2\}$ or $\{3,4\}$ as shown in Fig. 7. Both pairs $\{1,2\}$ and $\{3,4\}$

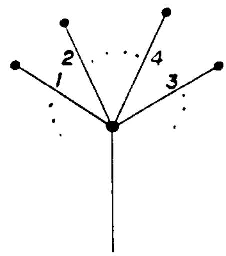


Fig. 7.

cannot occur as adjacent labels of adjacent outermost edges, for it follows from Claims 4 and 5 that at most two labels can occur on outermost edges.

Assume then, that only the pair $\{1, 2\}$ occurs as adjacent labels of adjacent outermost edges at the fork. This means that the outermost edges of the fork must contradict the conclusion of Claim 4, so the tunnel must connect separate components of γ_B. Then by Claim 5 we can assume, with no loss of generality that

i) every outermost edge is labelled $(1, 1)$ or $(1, 2)$, since $(1, 1)$ and $(2, 2)$ cannot both occur,

ii) at least one outermost edge at the fork is labelled $(1, 2)$,

iii) if not all are labelled $(1, 2)$ then every arc of $\partial E \cap S$ which has one end in ρ_2 must have the other end in ρ_1, for if both ends lie in ρ_2 the arc can be isotoped in S to a subarc of ρ_2, reducing the number of components of $E \cap S$.

On the other hand, if an outermost edge is labelled $(1, 1)$, there is an arc of $\partial E \cap S$ with both ends in ρ_1, so ∂E intersects ρ_1 more often than ρ_2. But ρ_1 and ρ_2 bound an annulus in $\partial W \cap A$; since the number of components in $E \cap S$ is minimal, this is impossible.

Thus all outermost edges at the fork are labelled $(1, 2)$. As in the proof of Claim 6 of 1.1, this is impossible. $\square$

Proof of 2.3. Since T has no outermost forks it must be a single edge or a point. The former case contradicts claim 2 and the remark which precedes it. In the latter case $S \cap E = \emptyset$, so E lies entirely in B. Remove a regular neighborhood of E from W; the result is a solid torus W'. Since $\pi_1(W') \simeq Z$ and $\pi_1(S) \to \pi_1(W' \cap A) = \pi_1(W \cap A)$ is injective, the inclusion induced $\pi_1(S) \to \pi_1(W' \cap B)$ fails to be injective. But a compressing disk for S in $W' \cap B$ will separate the two strands of γ_B in $B - \gamma_B \supset W'$, a contradiction. $\square$

3. Using outermost arcs to simplify the planar surface

The goal of this section is to use the standard 'outermost arc' trick of combinatorial three-dimensional topology to simplify the planar surface and its relation to the handlebody complement of the tunnel as much as possible.

3.1 Lemma. *Let $\gamma \subset S^3$ be a tunnel number one knot with tubular neighborhood $\gamma \times D^2$. Suppose there is an incompressible ∂-incompressible planar surface in $S^3 - (\gamma \times \mathring{D}^2)$ whose boundary is an odd number of longitudes in $\gamma \times \partial D^2$. Then such a planar surface can be found disjoint from a tunnel for γ.*

Proof. Let P be a planar surface in $S^3 - (\gamma \times \mathring{D}^2)$ whose boundary is $2k+1$ longitudes of $\gamma \times \partial D^2$. Let M be the manifold obtained from S^3 by zero-framed surgery on γ, i.e. by removing $\gamma \times D^2$ and sewing it back in with meridians and longitudes interchanged. Denote by $\bar{\gamma}$ the corresponding knot in M. Since $H_1(M) \simeq Z$, M contains no Lens space summands. Also, ∂P in M consists of $2k+1$ meridians of $\bar{\gamma} \times \partial D^2$; attaching the corresponding meridional disks turns P into a 2-sphere $\bar{P}$ and makes $\bar{\gamma}$ a $(2k+1)$-string composite in M.

Let $(I, \partial I) \subset (S^3 - (\gamma \times \mathring{D}^2), \gamma \times \partial D^2)$ be a PL imbedded arc (the tunnel) whose relative regular neighborhood has a complement a handlebody. Since $(S^3 - (\gamma \times \mathring{D}^2), \gamma \times \partial D^2) \simeq (M - (\bar{\gamma} \times \mathring{D}^2), \bar{\gamma} \times \partial D^2)$, Lemma 1.1 is applicable; there is an incompressible, boundary incompressible planar surface P' in $M - (\bar{\gamma} \times \mathring{D}^2) \simeq S^3 - (\gamma \times \mathring{D}^2)$, disjoint from I, whose boundary is isotopic in $\bar{\gamma} \times \partial D^2$ to $(2l+1)$ meridians, $l \leq k$. Perform the isotopy on $\bar{\gamma} \times \partial D^2$ and extend to an ambient isotopy of $M - (\bar{\gamma} \times \mathring{D}^2) \simeq S^3 - (\gamma \times \mathring{D}^2)$. Then the isotopy carries P' and I to the required disjoint planar surface and tunnel. $\square$

Following 3.1, we suppose γ is a tunnel number one knot in S^3 with tubular neighborhood $\gamma \times D^2$, $(I, \partial I) \subset (S^3 - (\gamma \times \mathring{D}^2), \gamma \times \partial D^2)$ is a tunnel for γ and $P \subset S^3 - (\gamma \times \mathring{D}^2)$ is a properly imbedded, boundary incompressible, incompressible planar surface, disjoint from I, such that ∂P consists of $(2k+1)$ longitudes of $\gamma \times \partial D^2$.

Label the boundary components $\{p_0, \dots, p_{n-1}\}$, $n = 2k+1$, of ∂P sequentially so that p_i and p_j, $i, j \in \mathbb{Z}_n$, cobound an annulus $A_{i,j}$ of $(\gamma \times \partial D^2) - \partial P$ if and only if $i = j-1$. Further choose the labelling so that one end of the tunnel lies in $A_{0,1}$. The other end then lies in an annulus $A_{r,r+1}$. Throughout the paper, $n \in \mathbb{Z}$ will refer to the number of boundary components and $r \in \mathbb{Z}_n$ will be defined as above. Furthermore, when r is referred to as an integer, it will denote the preimage of r in $\mathbb{Z}$ such that $0 \leq r \leq n-1$.

3.2 Remarks. There is a certain helpful ambiguity in this definition. In particular, if we label the components in the reverse order, so that p_i is relabelled p_{1-i}, for all $i \in \mathbb{Z}_n$ then $A_{r,r+1}$ becomes $A_{-r,-r+1}$ so, with the new relabelling, we can use $n-r$ for r.

The proof now splits into two distinct cases, $r = 0$ and $r \neq 0$. A combinatorial proof for case $r = 0$ can be constructed which is analogous to the proof for $r \neq 0$ which we will present. Jim Hoste, however, has suggested the following simpler argument.

3.3 Proposition. *If $r=0$ then γ is trivial.*

Proof. Lambert has proven the following proposition [4, Lemma 2].

If W is a 3-manifold in S^3 and P is a connected planar surface properly imbedded in W such that the number of components of ∂P is odd and each component of ∂P lies in an annulus $Y \subset \partial W$ and is parallel to the centerline of the annulus, then the centerline lies in the intersection of the entire lower central series of $\pi_1(W)$.

For the application we take W to be the closed complement of the union of $\gamma \times D^2$ and a regular neighborhood of the tunnel and Y to be $(\gamma \times \partial D^2) - (A_{0,1})$. Then a longitude of the knot lies in the intersection of the lower central series of $\pi_1(W) \cong Z * Z$, so is trivial. By the loop theorem, the knot is unknotted. □

Henceforth assume, then, that $r \neq 0$.

Choose a relative regular neighborhood of I intersecting $A_{0,1}$ in a cell I_0 and $A_{r,r+1}$ in a cell I_1. Let W be the closed complement of the union of the neighborhood and $\gamma \times D^2$. Then ∂W is a standard Heegaard splitting of S^3. As a result there is a 2-sphere in S^3 which intersects ∂W in a single essential circle. Among all such spheres transverse to $P \cup I_0 \cup I_1$, choose S to have a minimal number of components of intersection with $P \cup I_0 \cup I_1$. Let $E = S \cap W$ and $F = S - \mathring{W}$ be the two disks into which S is split by ∂W. By definition of S, all components of $E \cap P$ and $F \cap (I_0 \cup I_1)$ are arcs.

On $\partial E = \partial F$ label all points of intersection with $\partial P \cup \partial I_0 \cup \partial I_1$ as follows: A point in $\partial E \cap \partial p_i$ is labelled i, a point in $\partial E \cap \partial I_0$ is labelled a and a point in $\partial E \cup \partial I_1$ is labelled b. As we read the labels around $\partial E = \partial F$, it follows from the continuity of the attaching map of ∂E and the minimality of components of intersection of S with $P \cup I_0 \cup I_1$ that the labels around ∂E describe a path in the 'train track' shown in Fig. 8.

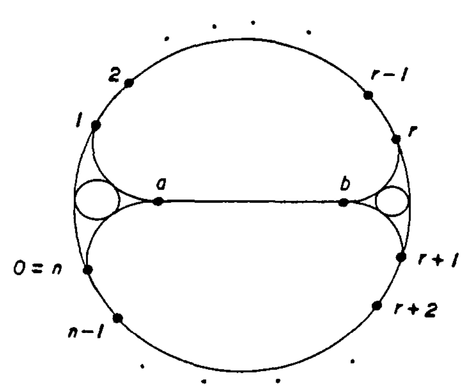


Fig. 8.

Actually, the truth is even simpler, for we have

3.4 Lemma. *No two points of $\partial E \cap (\partial I_0 \cup \partial I_1 \cup \partial P)$, adjacent in ∂E, are both labelled a or both labelled b.*

Proof. Suppose that β is an arc of $\partial E = \partial F$ on $(\gamma \times \partial D^2) - (I_0 \cup I_1)$ with both ends on the same component, say ∂I_0, of its boundary. Then β, together with an arc α in I_0, constitute a circle in $\gamma \times \partial D^2$.

Claim 1. *This circle cannot bound a cell in $\gamma \times \partial D^2$.*

Proof of Claim 1. If it did, then the cell must contain I_1, for otherwise β could be isotoped across ∂I_0, reducing the number of components of $I_0 \cap F$. The same argument applied to I_1 then shows that no arc of ∂E in $(\gamma \times \partial D^2) - (I_0 \cup I_1)$ can have both ends on ∂I_1. But then ∂E would intersect ∂I_0 more often than ∂I_1, which, since ∂I_1 and ∂I_0 are parallel in ∂W, would again allow a reduction in $\#(\partial E \cap \partial I_0)$. □

Now consider an outermost arc α of $F \cap (I_0 \cup I_1)$ in F. The arc α lies in, say, I_0. It, together with an arc β of $\partial E = \partial F$, bound a cell in either $\gamma \times D^2$ or the neighborhood of the tunnel. The latter is again impossible by minimality; it follows then from Claim 1 that $\alpha \cup \beta$ is a meridian.

Claim 2. *There can be no* arc η *of ∂E in $(\gamma \times D^2) - (I_0 \cup I_1)$ with both ends on the same component of $I_0 \cup I_1$ and disjoint from a longitude.*

Proof of Claim 2. This is clear if $\partial \eta \subset \partial I_1$, for in that case $I_1 \cup \eta$ would contain a longitude disjoint from the meridian $\alpha \cup \beta$. On the other hand, if $\partial \eta \subset \partial I_0$, then the complement of $I_0 \cup \eta \cup \beta$ in $\gamma \times \partial D^2$ would be a cell containing I_1. Then by Claim 1, every arc of ∂E with one end in ∂I_1 must have the other end in ∂I_0, implying, as in the proof of that claim, that ∂E intersects ∂I_0 more often than ∂I_1, and presenting the same contradiction. □

Proof of 3.4. The occurrence of the pattern (a, a) or (b, b) either allows a reduction of $\#(\partial E \cap (\partial I_0 \cup \partial I_1))$ or implies the existence of an arc η as in Claim 2. □

3.5 Lemma. *If two points of $\partial E \cap (\partial I_0 \cup \partial I_1 \cup \partial P)$, adjacent in ∂E, are both labelled 1 or both labelled r (resp. both $r+1$ or both n) then no occurrence of $r+1$ or n (resp. 1 or r) is adjacent to an occurrence of a, b, or itself.*

Proof. Suppose, for example, that two adjacent points in ∂E are both labelled 1. Then the arc β between them in ∂E, together with an arc of p_1 bound a disk in $\gamma \times \partial D^2$. By the minimality of $\#\partial E \cap (\partial I_0 \cup \partial I_1 \cup \partial P)$, this disk must contain I_0.

Hence no occurrence of a can be adjacent to an occurrence of n, nor can two occurrences of n be adjacent. Let $m_a = \#(\partial E \cap \partial I_0)$, $m_b = \#(\partial E \cap \partial I_1)$ and $m_i = \#(\partial E \cap p_i)$ $i \in \mathbb{Z}_n$. Then it follows that $m_1 \geq m_n + m_a + 2$. Furthermore, since the pairs p_1 and p_r, ∂I_1 and ∂I_0 and p_{r+1} and p_n each cobound an annulus in ∂W, minimality requires that $m_1 = m_r$, $m_a = m_b$, $m_{r+1} = m_n$. If there were no adjacent occurrences of r, then $m_{r+1} \geq m_r - m_b$, or $m_n + m_a \geq m_1$, contradicting the previous inequality. Thus there are also adjacent occurrences of r, hence no adjacent occurrences of $r+1$ nor an occurrence of $r+1$ adjacent to one of b. The lemma then follows from Fig. 1. □

Remark. It follows from Fig. 8, 3.4 and 3.5 that the labels around ∂E describe a path in one of the 'train tracks' shown in Fig. 9.

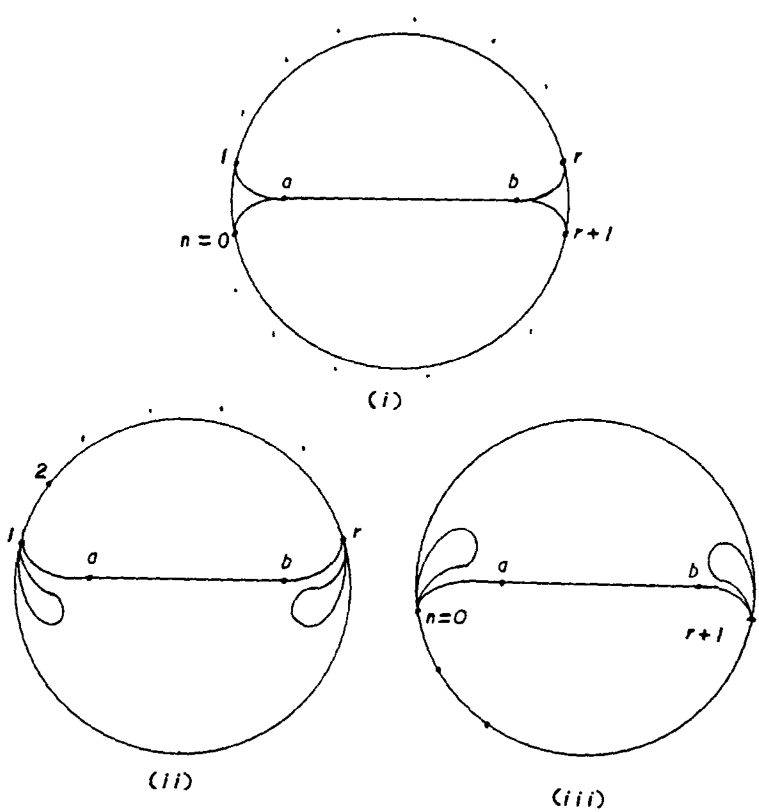


Fig. 9.

3.6 Lemma. *Each outermost arc of $P \cap E$ in E is labelled with one of the pairs*

$$(0, r), (0, r+1), (1, r), (1, r+1).$$

Proof. Let α be an outermost arc of $P \cap E$ in E, and β be the arc of ∂E which, together with α, bounds the outermost cell in $E - P$. From 3.4 it suffices to show

Claim. *The label a or b appears on the interior of* β.

Proof of claim. If not, β lies in $\gamma \times \partial D^2$. Since P is boundary incompressible in $S^3 - (\gamma \times \mathring{D}^2)$ this would require that for some arc η of ∂P, $\alpha \cup \eta$ bounds a cell in P. Then $\beta \cup \eta \subset \gamma \times \partial D^2$ bounds a cell Q in W which can be isotoped to be disjoint from P and I. Then, since γ is not trivial, ∂Q bounds a cell Q' in $\gamma \times D^2$. If Q' contains neither I_0 nor I_1, then β could be pushed across Q', reducing the number of components of $E \cap P$. On the other hand, if either I_0 or I_1 is in Q', push int(Q') slightly into $\gamma \times D^2$. Then $Q \cup Q'$ becomes a sphere intersecting ∂W in an essential circle, but disjoint form $P \cup I_0 \cup I_1$, contradicting our definition of S. □

3.7 Lemma. *Not all the outermost arcs of* $P \cap E$ *in* E *are labelled with the pair* $(1, r)$ *nor all with the pair* $(r+1, n)$.

Proof. The proof is an exact duplicate of the proof of Claims 4 through 9 of Lemma 1.1 (the 'outermost fork' argument).

3.8 Remark. It follows from 3.5, 3.6 and 3.7 that the labels around ∂E in fact lie in a path in train track (i) of Fig. 9. In particular, if there is a series of labels appearing on ∂E which begins with either the pattern $a, 1, 2, \ldots$ or with $a, n, n-1, n-2, \ldots$ and ends with either $\ldots, n-2, n-1, n, a$ or $\ldots, 2, 1, a$ then the series must contain within it, adjacent and in order, all of $\mathbb{Z}_n$ (either $\ldots 1, 2, 3, \ldots n-2, n-1, n, \ldots$ or $\ldots n, n-1, n-2, \ldots, 2, 1, \ldots$).

Similarly if there is a series of labels appearing on ∂E which begins with either $b, r+1, r+2, \ldots$ or with $b, r, r-1, \ldots$ and ends with either $\ldots r-1, r, b$, or with $\ldots r+2, r+1, b$, then the series must contain within it either the series $r+1, r+2, \ldots, n, 1, \ldots, r$ or the series $r, r-1, \ldots, 1, n, \ldots, r+1$. This fact will be crucial in Section 5.

4. Bilabelled trees

A bilabelled tree will be a finite tree, linearly embedded in the plane, with each side of each edge assigned an integer (the *label*). Thus each edge is assigned two labels; if two labels are assigned to the same edge somewhere in the tree we say that the labels are *dual*.

Two edges e_0, e_1 in a bilabelled tree T are said to be *adjacent* if e_0, e_1 share a common vertex v and, in any neighborhood U of v there is a PL arc $\alpha : (I, \partial I) \to (U, U \cap T)$ such that $\alpha^{-1}(T) = \partial I$, and $\alpha(i) \in e_i - v$, $i = 0, 1$. The sides of e_0 and e_1 on which α lies will be called *adjacent sides* of the adjacent edges.

In general, a finite tree in the plane can be deformed so that it lies in a disk D with $\partial D \cap T$ exactly the ends of T. The boundary of a component C of $D - T$ then consists of an arc in ∂D together with an imbedded path P in T. The path P, with each edge assigned the (single) label which lies in C is called an *end-path* of T. If T has n ends, it has n end-paths.

An edge of T is called *neutral* if the labels on both sides are equal. A vertex is called neutral if it is not an end of T and the edges which abut it are labelled (as shown in Fig. 10) so that always $a_i \leqslant b_i$ or always $a_i \geqslant b_i$.

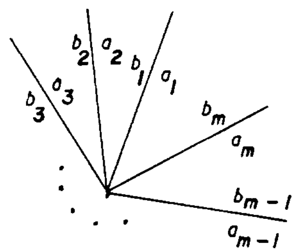


Fig. 10.

4.1 Definition. For P and P' end-paths of T, define $P \leqslant P'$ if there is a series of end-paths $P = P_1, \ldots, P_m = P'$ such that for each $i = 2, \ldots, m$, $P_{i-1} \cap P_i$ contains an edge whose label in P_{i-1} is less than or equal to its label in P_i.

4.2 Lemma. *If T has no neutral simplices, $\leqslant$ is a partial order on the end-paths of T.*

Proof. The difficulty is to show that $P \leqslant P'$ and $P' \leqslant P$ implies $P = P'$ (antisymmetry).

Suppose there are distinct end-paths $P_1, \ldots, P_m$ such that for every $i \in \mathbb{Z}_m$, $P_{i-1} \cap P_i$ contains an edge e_i such that the label of e_i in P_{i-1} is less than or equal to that in P_i. For each $i \in \mathbb{Z}_m$ there is a path α_i in P_i from e_i to e_{i+1}; connect one end of each α_{i-1} to α_i in e_i to form a map $\alpha : S^1 \to T$.

Call such a collection $\{P_i\}$ of end-paths a *cycle* of end-paths, and α an associated loop in T.

Let α be the shortest loop in T associated to any cycle of end-paths. That is, of all loops associated to all cycles of end-paths, the image of α contains the fewest edges.

Claim. *The image of α contains no edges.*

Proof of claim. Suppose that the image of α contains the edge e. Then, by construction, e is in two edge paths, say P_1 and P_i, in a cycle of edge paths $P_1, \ldots, P_m$. With no loss of generality assume the label of e in P_1 is less than or equal to that of e in P_i. Then the collection $\{P_1, \ldots, P_i\}$ is also a cycle of end-paths with an associated loop of length $\leqslant \alpha$. Thus we can assume $i = m$ and $e = e_1$. Now the only end-paths containing $e = e_1$ are P_m and P_1, so every α_i, $1 < i < m$, lies in the same component of the graph $T - (\operatorname{int} e)$. Then we can choose for α_1 and α_m arcs which meet at the vertex of ∂e contained in that component, reducing the length of α. $\square$

Proof of Lemma 4.2. It follows from the claim that if $\{P_1, \ldots, P_m\}$ is a cycle of end-paths, then $e_1, \ldots, e_m$ must all abut on the same vertex. Since any edge is in exactly two end-paths, if $m \geqslant 3$ the e_i are all distinct and, for $i \in \mathbb{Z}_m$, each e_{i-1} is adjacent to e_i. The vertex is then neutral, contradicting the hypothesis. If $m = 2$ and $e_1 \neq e_2$ the same argument applies.

If $m = 2$ and $e_1 = e_2$, then $e_1 = e_2$ is a neutral edge again contradicting hypothesis.

Hence there are no cycles of end-paths. Antisymmetry, hence the lemma, is then immediate. □

We are interested in a particular bilabelled tree that arises from the discussion in Section 3. As in Section 3.1, let M be the manifold obtained by 0-framed surgery on γ. Let $\tilde{M}$ be the infinite cyclic cover of M, $\tilde{\gamma} \times D^2 \subset \tilde{M}$ the infinite cyclic cover of $\bar{\gamma} \times D^2$, and $\tilde{P} \subset \tilde{M}$ the associated cover of P. Then $\tilde{P}$ consists of infinitely many planar surfaces. Label the components $\{\tilde{p}_i\}$ of $\partial \tilde{P} = \tilde{P} \cap (\tilde{\gamma} \times \partial D^2)$ in order along $\tilde{\gamma}$ so that the label corresponds mod n to the label of its projection in $M - (\bar{\gamma} \times \mathring{D}^2) = S^3 - (\gamma \times \mathring{D}^2)$ described in Section 3.

There is a fixed $t \in \mathbb{Z}$ such that any lift of the tunnel in $\tilde{M}$ runs between annu'i of $(\tilde{\gamma} \times \partial D^2) - \tilde{P}$ bounded by components of $\partial \tilde{P}$ labelled $\{kn, kn+1\}$ and $\{(k+t)n + r, (k+t)n + r + 1\}$, some $k \in \mathbb{Z}$. By use of the symmetry described in 3.2 we can assume that $t \geqslant 0$. Moreover, since each component of $\tilde{P}$ separates $\tilde{M}$, the cycle composed of an arc in $\tilde{\gamma}$ from $\tilde{p}_{kn+1}$ to $\tilde{p}_{(k+t)n+r}$ together with an arc across the tunnel between the same components must intersect $\tilde{P}$ an even number of times. Therefore $tn + r$ is positive even.

Let $\tilde{E}$ be a fixed lift of the disk E of Section 3 to $\tilde{M} - (\tilde{\gamma} \times \mathring{D}^2)$. Then $\tilde{E} \cap \tilde{P}$ is a collection of arcs in $\tilde{E}$, each of whose ends we label with the corresponding label of the component of $\partial \tilde{P}$ in which it lies. Associated to $\tilde{E} \cap \tilde{P}$ is a bilabelled tree T obtained by choosing a vertex in every component of $\tilde{E} - \tilde{P}$, and an edge between vertices corresponding to adjacent components of $\tilde{E} - \tilde{P}$. Such an edge e crosses a single component α of $\tilde{E} \cap \tilde{P}$ and we assign the label of an end of α to the side of e on which it lies.

4.3 Remark. It follows from 3.6 that for each outermost edge of T there is a $k \in \mathbb{Z}$ such that the labels of the edge are one of these four pairs:

$$(kn, (k+t)n + r + 1)$$

$$(kn, (k+t)n + r)$$

$$(kn + 1, (k+t)n + r + 1)$$

$$(kn + 1, (k+t)n + r).$$

It also follows from Fig. 9 that adjacent labels of adjacent edges differ by 1, except for the four pairs above which arise precisely when $\partial \tilde{E}$ crosses a lift of the tunnel.

4.4 Proposition. *At a neutral vertex of T there are either*

a) *at least two neutral edges abutting on the vertex, or*

b) *two adjacent edges abutting on the vertex such that, for some* $k \in \mathbb{Z}$, *their adjacent labels are one of the four pairs listed in* 4.3, *and represent a crossing of the tunnel. Furthermore if* a *is the dual of* kn (*resp.* $kn+1$) *on one of these two edges, then* $a \geq kn$ (*resp.* $kn+1$) *and if* b *is the dual of* $(k+t)n+r$ (*resp.* $(k+t)n+r+1$) *then* $b \leq (k+t)n+r$ (*resp.* $(k+t)n+r+1$).

Proof. Suppose the neutral vertex is labelled as shown in Fig. 10, with $b_i \geq a_i$, $i \in \mathbb{Z}_m$, $m > 1$. From Remark 4.3 we see that if b) is not true at this vertex then the inequality $a_i + 1 \geq b_{i-1}$ is true for every $i \in \mathbb{Z}_m$. Consequently there is the following series of inequalities

$$a_1 \leq b_1 \leq a_2 + 1 \leq b_2 + 1 \leq a_3 + 2 \leq \cdots \leq b_m + (m-1) \leq a_1 + m. \qquad (*)$$

Then at least m of these $2m$ inequalities must in fact, be equalities. Consider the cases:

Case i) No equality of the form $a_i + (i-1) = b_i + (i-1)$ occurs in $(*)$.

Then $b_{i-1} = a_i + 1$, and $b_i = a_i + 1$ for every $i \in \mathbb{Z}_m$, so each $a_i = a_1$ and each $b_i = a_1 + 1$. Denote a_1 by q and consider the cell F in $\tilde{E} - \tilde{P}$ corresponding to the vertex (Fig. 11). It has boundary (oriented counterclockwise) consisting of arcs running from $\tilde{p}_q$ to $\tilde{p}_{q+1}$ in $\tilde{P}$ and from $\tilde{p}_{q+1}$ to $\tilde{p}_q$ in the annular segment of $\tilde{\gamma} \times \partial D^2$ which lies between $\tilde{p}_q$ and $\tilde{p}_{q+1}$. (If the latter arc crossed a tunnel we would be in case b) of the proposition.) Then $\tilde{p}_q$ and $\tilde{p}_{q+1}$ lie in the same component of $\tilde{P}$. Complete this component to a sphere $\bar{P}$ lying in $\tilde{M}$ by attaching meridians of $\tilde{\gamma} \times D^2$ to its boundary. To a collar of $\bar{P}$ attach further the segment of $\tilde{\gamma} \times D^2$ lying between $\tilde{p}_q$ and $\tilde{p}_{q+1}$. The result is a punctured solid torus in $\tilde{M}$; F is a 2-cell in its complement that intersects a meridian of the solid torus algebraically m times. Thus, attaching a relative regular neighborhood of F gives a twice punctured Lens space in $\tilde{M}$. But $\tilde{M}$ can contain no Lens space summands (e.g. because its homology is finitely generated over the field $\mathbb{Z}_q$, q a prime factor of m [5]).

Case ii) Precisely one equality of the form $a_i + (i-1) = b_i + (i-1)$ and one inequality of the form $b_j + (j-1) < a_{j+1} + j$ appear in $(*)$.

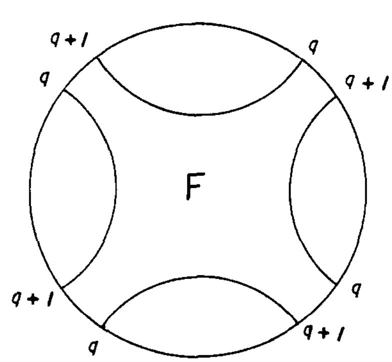


Fig. 11.

Then there are m inequalities, so each must denote a difference of precisely one. Then $b_j = a_{j+1}$, contradicting Remark 4.3.

Case iii) Precisely one equality of the form $a_i + (i-1) = b_i + (i-1)$ appears in (*) and, for all $j \in \mathbb{Z}_m$, $b_j + (j-1) = a_{j+1} + j$.

With no loss of generality, let $i = 1$. There are $(m+1)$ equalities in (*) so $(m-1)$ strict inequalities. Then $(m-2)$ of these must come from a difference of exactly 1, while one may arise from a difference of 2. Thus there is a $j \neq 1$ such that $a_j = b_j - 1$ or $b_j - 2$ and for $1 \neq i \neq j$, $a_i = b_i - 1$. The possibility $a_j = b_j - 1$ cannot occur, for it would produce the parity contradiction:

$$a_1 \equiv b_1 \not\equiv a_2 \not\equiv b_2 \not\equiv \cdots \not\equiv b_n \not\equiv a_1 \pmod 2.$$

Thus $a_j = b_j - 2$ and the vertex is labelled as shown $(q = a_1)$. Consider the arc (Fig. 12) of $E \cap P$ corresponding to the neutral edge labelled (q, q).

In $\tilde{P}$ this is an arc lying in a component of $\tilde{P}$, both of whose ends lie on $\tilde{p}_q$. The dotted line in Fig. 12 is an arc in a small bicollar of $\tilde{P}$ whose ends terminate in the annulus of $\tilde{\gamma} \times \partial D^2$ which lies between $\tilde{p}_q$ and $\tilde{p}_{q-1}$.

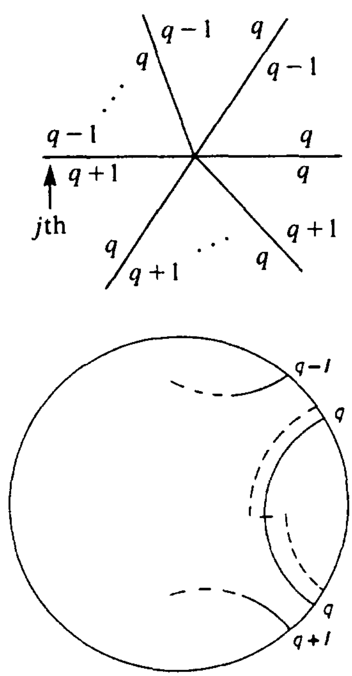


Fig. 12.

(Again, if, instead one of the arcs in ∂E from q to $q-1$ or q to $q+1$ crossed a lift of the tunnel we would be in case b) of the proposition.) The ends can be connected in that annulus to produce a circle in the bicollar of $\tilde{P}$ which intersects $\tilde{P}$ once. Since $\tilde{P}$ is two-sided in $\tilde{M} - (\tilde{\gamma} \times D^2)$ this is impossible.

Since there are at least m equalities in $(*)$ all that remains is

Case iv) There is more than one equality of the form $a_i+(i-1)=b_i+(i-1)$ appearing in $(*)$.

This is case a) of the proposition. $\square$

5. The Poenaru conjecture

In this section we complete the proof of

5.1 Theorem. *Let γ be a knot in S^3 with tubular neighborhood $\gamma \times D^2$. Suppose there is a compressible, boundary incompressible planar surface $(P, \partial P) \subset (S^3-(\gamma \times \mathring{D}^2), \partial D^2)$ such that ∂P has $2k+1$ components. Then the tunnel number of γ is not one.*

Proof. Let E be the properly imbedded disk in $S^3-(\gamma \times \mathring{D}^2)$, defined in Section 3, which separates $S^3-(\gamma \times \mathring{D}^2)$ into two solid tori. Choose a tunnel disjoint from P (or similar planar surface) as in Section 1.

Let $\tilde{M}$, $\tilde{P}$, $\tilde{E}$ and T be defined as in Section 4 and suppose T is not a vertex (i.e. $P \cap E \neq \emptyset$). Call a simplex s an *outermost neutral simplex* if all but one of the components of $T-s$ contain no neutral simplices. We intend to derive a contradiction to the assumption $P \cap E \neq \emptyset$ by examining such a simplex.

Case i) T contains no neutral simplices.

T contains an end-path Q which is minimal under the partial ordering of 4.2. Since Q is minimal it follows from 3.6 and 4.3 that, for some $k, l \in \mathbb{Z}$, one end of Q begins its labelling with $kn+1, kn+2, \ldots$ or $kn, kn-1, \ldots$ while the other end terminates with the labelling $\ldots ln-1, ln$ or $\ldots ln+2, ln+1$. Interpreting 3.5 in the infinite cyclic cover, this means that Q contains an entire series of the form $jn+1, jn+2, \ldots, (j+1)n$ or $(j+1)n, (j+1)n-1, \ldots, jn+1$ for some $j \in \mathbb{Z}$. Thus each of these labels is dual to a higher label. By applying covering translations it follows that every label is dual to a higher label. Now to say label j is dual to label k is to imply there is an arc in $\tilde{P}$ from $\tilde{p}_j$ to $\tilde{p}_k$. Hence we conclude that for any $\tilde{p}_j$ there is an arc in $\tilde{P}$ connecting $\tilde{p}_j$ to $\tilde{p}_k$ for arbitrarily high values of k. Since each component of $\tilde{P}$ is compact, this is impossible.

(For future reference: the above argument also works for the *maximal* end-path of Q.)

Case ii) T contains an outermost neutral simplex s which is an edge.

By 4.3 s is not an outermost edge of T. By definition, one of the trees T' obtained by removing $(\operatorname{int} s)$ from T contains no neutral simplices. Moreover, because s is neutral while the vertex $v = s \cap T'$ is not neutral in T, it follows that v is neither an end nor a neutral vertex of T'. Thus T' has no neutral simplices, yet all but one end-path of T' is also an end-path of T. Hence either a maximal or minimal end-path of T' under the partial order $\leqslant$ of 4.2 lies in T. This end-path produces the same contradiction as in case i).

Case iii) T contains an outermost neutral simplex which is a vertex v.

Apply 4.4. Note that option a) of 4.4 cannot apply at v, or v would not be an outermost neutral simplex. All but one of the trees obtained by splitting T at v (i.e. taking the closure of each component of $T-v$) contains no neutral simplices. By 4.4 b) one of them, say T', has the end at v labelled with one of the patterns

$$v\ \bullet\!\frac{\quad a\quad}{kn}\!\bullet\!-\cdots \qquad v\ \bullet\!\frac{\quad a\quad}{kn+1}\!\bullet\!-\cdots \qquad v\ \bullet\!\frac{\qquad b\qquad}{(k+t)n+r}\!\bullet\!-\cdots$$

$$v\ \bullet\!\frac{\qquad b\qquad}{(k+t)n+r+1}\!\bullet\!-\cdots$$

where a is greater than and b less than its dual in the diagram. In the first two cases let Q be an end-path for T' minimal under the partial ordering $\leqslant$ or, in the last two cases, let Q be an end-path for T' maximal under the partial ordering $\leqslant$. Then argue, as in *case i*), that either every label is dual to a higher label or every label is dual to a lower label, and derive the same contradiction.

Hence we conclude that $E \cap P = \emptyset$. Consider, then, the closure C of the component of $S^3-((\gamma\times D^2)\cup E)$ in which P lies. C is, by definition of E, a solid torus, and ∂P consists of $(2k+1)$ circles in its boundary. Either all of these circles are parallel in ∂C, hence null-homologous in C, hence meridians, or one of the circles is inessential in ∂C. In either case a component of ∂P bounds a disk in C, hence also a disk in $S^3-(\gamma\times \mathring{D}^2)$. Therefore γ is trivial. $\square$

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