Homework 3 Hints

Problems from Crossley’s Book

4.1 All that you need is the definition of connected (and disconnected) space. A proof by contradiction would be efficient.

4.8 To warm up, prove the problem for the \( n = 3 \) case. Then figure out how to give a proof by induction for the general case.

By definition of Hausdorff space, there are neighborhoods \( U_1, U_2 \) of \( x_1, x_2 \), respectively, so that \( U_1 \cap U_2 = \emptyset \); so the problem is solved for the \( n = 2 \) case.

To extend this to the \( n = 3 \) case, use the definition of Hausdorff space to get disjoint neighborhoods of \( x_1 \) and \( x_3 \); do this again for the pair \( x_2 \) and \( x_3 \). Then form various intersections to obtain a collection of open sets \( V_1, V_2, V_3 \), each containing one, and exactly one, of the points \( x_1, x_2, x_3 \).

4.10 A little formality will be useful here. Let \( L = \mathbb{R} \cup \{0'\} \), and consider the function \( p : L \to \mathbb{R} \) defined by the formulas \( p(0') = 0 \), and \( p(t) = t \) if \( t \in \mathbb{R} \).

Then \( U \subset L \) is open if \( p(U) \subset \mathbb{R} \) is open (in the usually topology). The problem requires you to show that the collection

\[ T_L = \{ U \subset L : p(U) \subset \mathbb{R} \text{ is open} \} \]

is a topology on \( L \). This means that you need to show that \( \emptyset \in T_L \), \( L \in T_L \), arbitrary unions of elements of \( T_L \) are in \( T_L \), and finite intersections of elements of \( T_L \) are in \( T_L \).
You may use the following facts from basic set theory (without proof): Let $f : X \to Y$ be a function. Then

\[- f(\bigcup O_\alpha) = \bigcup f(O_\alpha) \text{ for any collection } \{O_\alpha\} \text{ of subsets of } X.\]

\[- f(A \cap B) \subset f(A) \cap f(B) \text{ for any pair of subsets } A \text{ and } B \text{ of } X.\]

The first fact about unions will help you to show that arbitrary unions of elements in $T_L$ are in $T_L$.

The second fact will be useful for finite intersections. Keep in mind that the function $p : L \to \mathbb{R}$ is very special: it is the identity function except at $0$ and $0'$. You are recommended to prove the following

**Lemma.** For any two subsets $A, B \subset L$, we have

\[p(A \cap B) = p(A) \cap p(B) \text{ or } p(A \cap B) = (p(A) \cap p(B)) - \{0\}.\]

Then figure out a way to apply this lemma to show that if $A, B \in T_L$, then $A \cap B \in T_L$.

**Problems from Hatcher’s Notes**

15(b) Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is **continuous at $x \in \mathbb{R}$ on the right** if and only if $\lim_{\epsilon \to 0^+} f(x + \epsilon) = f(x)$.

This means that for every $\epsilon' > 0$, there exists a $D > 0$, so that $|f(x + \epsilon) - f(x)| < \epsilon'$ for every $0 < \epsilon < D$. To relate this to the “lower limit topology” (a.k.a. “half-open” topology) on $\mathbb{R}$, show that $\lim_{\epsilon \to 0^+} f(x + \epsilon) = f(x)$ means that for every $\epsilon' > 0$, there is a $D > 0$, so that $[x, x + D) \subset f^{-1}(f(x) - \epsilon', f(x) + \epsilon')$. 
