Review of Set Theory

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Basic Set Theory

We will use the standard notation for containments: if x is an element of a set A, then we write $x \in A$; otherwise we write $x \notin A$. If A is a subset of a set B, we will write $A \subset B$ or $A \subseteq B$; otherwise we write $A \not\subseteq B$. If $A \subset B$ but $A \neq B$, we could write $A \subsetneq B$ (to emphasize that $A \neq B$). Note that we always have $\varnothing \subset A$. If B is a set, a **proper subset** of B is a subset A for which $\varnothing \subsetneq A \subsetneq B$.

Definitions. Let A and B be sets.

- The **union** of A and B is the set $A \cup B = \{x : x \in A \text{ or } x \in B\}$.
- The **intersection** of A and B is the set $A \cap B = \{x : x \in A \text{ and } x \in B\}$.
- The **complement** of B in A is the set $A - B = \{x : x \in A, \text{ but } x \notin B\}$.
- The (ordered) **Cartesian product** of A with B is

  $$A \times B = \{(a, b) : a \in A \text{ and } b \in B\} .$$

  More generally, if $n \in \mathbb{N}$ is given and $X_1, \ldots, X_n$ are sets, we can form the (ordered) **Cartesian product** of $X_1, \ldots, X_n$ as

  $$X_1 \times \cdots \times X_n = \{(x_1, \ldots, x_n) : x_i \in X_i \text{ for } i = 1, \ldots, n\} .$$

  We call the X_i the **factors** of the product. In particular, if all of the factors are the same set X, we have the **n-fold product**

  $$X^n = \underbrace{X \times \cdots \times X}_{n \text{ factors}} .$$

For example, $\mathbb{R}^1$ is just $\mathbb{R}$, $\mathbb{R}^2$ is the x, y–plane and $\mathbb{R}^3$ is x, y, z–space.

Definition. Let A and B be sets (possibly with nonempty intersection). The **disjoint union** of A with B is the set

$$A \coprod B = \{(a,1) : a \in A\} \cup \{\,(b,2) : b \in B\}\ .$$

With this formal definition, we see that $A \coprod B \subset (A \cup B) \times \{1, 2\}$. The important thing to realize is that $A \coprod B$ *is the set formed by taking the elements of* A *and* B *separately; so any points of* $A \cap B$ *will be counted twice* in $A \coprod B$. This notion of disjoint union will be especially important later on when we study *quotient spaces*.

One can use parentheses to form new sets: if A, B and C are sets, we can form the set $A \cap (B \cup C) = \{x : x \in A \text{ and } x \in B \cup C\} = \{x : x \in A \text{ and } [x \in B \text{ or } x \in C]\}$. In general, $A \cap (B \cup C) \neq (A \cap B) \cup C$; for example, we have inequality when $A = \{1\}, B = \{2\}$ and $C = \{1, 2\}$. What is true, however, are the **distributive laws**:

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

As long as the set operations are all unions or all intersection, there is no trouble with moving parentheses (i.e. we have **associativity**). For example, $A \cup B \cup C = (A \cup B) \cup C = A \cup (B \cup C)$. We can unambiguously define the union or intersection of several (possibly infinitely many) sets.

Definitions. Given a set $\mathcal{J}$, suppose that to each $\alpha \in \mathcal{J}$ we assign a set A_α. We refer to $\mathcal{J}$ as the **index set** for the collection $\{A_\alpha\}_{\alpha \in \mathcal{J}}$. We can then form the **union**

$$\bigcup_{\alpha \in \mathcal{J}} A_\alpha = \{x : x \in A_\alpha \text{ for some } \alpha \in \mathcal{J}\}\ ,$$

and the **intersection**

$$\bigcap_{\alpha \in \mathcal{J}} A_\alpha = \{x : x \in A_\alpha \text{ for every } \alpha \in \mathcal{J}\}\ .$$

When the index set $\mathcal{J}$ is understood from context, we can just write $\bigcup A_\alpha$ for union, and $\bigcap A_\alpha$ for intersection.

A word about proofs

Suppose that we are given two sets X and Y and we assume that they have certain properties. How would we prove that $X = Y$. The typical way to prove that $X = Y$ is to separately prove that $X \subset Y$ and $Y \subset X$. To prove the first containment $X \subset Y$, we let x be an arbitrary element of X and prove that x must necessarily be an element of Y. This is the method of **element-chasing**.

As a first example, consider the distributive law $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ for sets A, B and C. A proof of the containment $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$ could go like this:

> Let $x \in A \cap (B \cup C)$. Then by definition of $A \cap (B \cup C)$, we have that $x \in A$ and $x \in B \cup C$. Hence $x \in A$ and $[x \in B$ or $x \in C]$. By definition of $A \cap B$ and $A \cap C$, we conclude that $x \in A \cap B$ or $x \in A \cap C$. Thus $x \in (A \cap B) \cup (A \cap C)$, as desired.

A symbolic version of the proof could go like this:

$$\begin{aligned} x \in A \cap (B \cup C) &\Rightarrow x \in A \text{ and } x \in B \cup C \\ &\Rightarrow x \in A \text{ and } [x \in B \text{ or } x \in C] \\ &\Rightarrow [x \in A \text{ and } x \in B] \text{ or } [x \in A \text{ and } x \in C] \\ &\Rightarrow x \in A \cap B \text{ or } x \in A \cap C \\ &\Rightarrow x \in (A \cap B) \cup (A \cap C). \end{aligned}$$

It is now clear that the implications need only be reversed to obtain a proof of the reverse implication $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$. Of course, this will not always be the case in set theory.

We can now give a succinct symbolic proof of $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ as follows:

$$\begin{aligned} x \in A \cap (B \cup C) &\Leftrightarrow x \in A \text{ and } x \in B \cup C \\ &\Leftrightarrow x \in A \text{ and } [x \in B \text{ or } x \in C] \\ &\Leftrightarrow [x \in A \text{ and } x \in B] \text{ or } [x \in A \text{ and } x \in C] \\ &\Leftrightarrow x \in A \cap B \text{ or } x \in A \cap C \\ &\Leftrightarrow x \in (A \cap B) \cup (A \cap C). \end{aligned}$$

Sometimes it is preferable to not give a symbolic version of the proof, because some explaining is in order. For example, here is a proof of the equality

$$(A - C) \cup B = (A \cup B) - (C - B) \ .$$

We show that

1. $(A - C) \cup B \subset (A \cup B) - (C - B)$, and
2. $(A \cup B) - (C - B) \subset (A - C) \cup B$.

For the first containment, let $x \in (A - C) \cup B$ be given. Then $x \in A - C$ or $x \in B$. Suppose that $x \in A - C$. Then $x \in A$ and $x \notin C$. Since $A \subset A \cup B$ and $C - B \subset C$, we conclude that $x \in A \cup B$ and $x \notin C - B$. Hence $x \in (A \cup B) - (C - B)$. Now suppose that $x \in B$. By definitions of $A \cup B$ and $C - B$, it is clear that $x \in A \cup B$ and $x \notin C - B$. Hence $x \in (A \cup B) - (C - B)$.

For the second containment, let $x \in (A \cup B) - (C - B)$ be given. Then $x \in A \cup B$ and $x \notin C - B$. For convenience, we divide the proof of $x \in (A - C) \cup B$ into two cases: $x \in B$ and $x \notin B$. Suppose that $x \in B$. This immediately implies that $x \in (A - C) \cup B$. Now suppose that $x \notin B$. Then $x \in A$ and $x \in C - B$. Since $x \notin C - B$ and $x \notin B$, we must have $x \notin C$. Thus $x \in A$ and $x \notin C$. Hence $x \in (A - C) \subset (A - C) \cup B$. This concludes the proof.

Functions

Definitions (Basic definitions)**.** Let $f : A \to B$ be a function.

- A is called the **domain** (or **source**) of f.
- B is called the **codomain** (or **target**) of f.
- If $X \subset A$, we define the **image** of X under f to be

$$f(X) = \{f(a) : a \in X\} \ .$$

- If $U \subset B$, we define the **preimage** of U under f to be

$$f^{-1}(U) = \{a : f(a) \in U\} \ .$$

- If $f' : A \to B$ is another function, then we write $f = f'$ whenever

$$f(a) = f'(a) \text{ for every } a \in A \ .$$

Definitions (Special functions)**.** Let $f : A \to B$ be a function.

- If $A = B$ and $f(a) = a$ for all $a \in A$, then f is called the **identity (function)** on A. In this case, f is usually denoted by id_A.

- If $A \subset B$ and $f(a) = a$ for all $a \in A$, then f is called the **inclusion (function)** of A into B.

Definitions (Injection, surjection and bijection)**.** Let $f : A \to B$ be a function.

- f is **injective** (or **one-to-one**) if

$$f(a_1) \neq f(a_2) \text{ whenever } a_1 \neq a_2 \ .$$

  In this case, f is called an **injection**.

- f is **surjective** (or **onto**) if $f(A) = B$. In this case, f is called a **surjection**.

- f is **bijective** if f is injective and surjective. In this case, f is called a **bijection** (or a **one-to-one correspondence**).

Definition (Equivalence (or isomorphism) of sets)**.** If A and B are sets, then they are **equivalent (as sets)** (or **isomorphic (as sets)**) if there exits a bijection $f : A \to B$. We will usually denote this by $A \cong B$. In this sort of language, we refer to a bijection from A to B as an **isomorphism** of the sets A and B.

Definition (Composition of functions)**.** Let $f : A \to B$ and $g : C \to D$ be functions. If $f(A) \subset C$, we can define the **composition of *f* followed by *g*** to be the function $g \circ f : A \to D$ given by the formula $g \circ f(a) = g(f(a))$. Furthermore, suppose that $h : E \to F$ is another function with $g(C) \subset E$; then, we can define the compositions

$h \circ (g \circ f) : A \to F$ and $(h \circ g) \circ f : A \to F$. It is easy to check that **associativity** (of compositions) holds, that is,

$$h \circ (g \circ f) = (h \circ g) \circ f \ .$$

We can now unambiguously express this composition as just $h \circ g \circ f : A \to F$.

Definition. Let $f : A \to B$ be a function. Suppose that there exists a function $g : B \to A$ that satisfies

$$g \circ f = \mathrm{id}_A \ \text{ and } \ f \circ g = \mathrm{id}_B \ .$$

Then f is called **invertible**, g is called an **inverse (function)** of f, and we may denote g by the symbol f^{-1}.

Proposition. *A function $f : A \to B$ is invertible if and only if it is a bijection. Furthermore, an invertible function has a unique inverse function.*

Proof. We prove that if f has an inverse function, then it is unique. The other part of the proposition is left as an exercise. Suppose that $g, g' : B \to A$ are both inverses of a function $f : A \to B$. We show that $g = g'$. Let $b \in B$ be given. Then

$$\begin{aligned} g(b) &= g(\mathrm{id}_B(b)) \\ &= g(f \circ g'(b)) && [\text{since } f \circ g' = \mathrm{id}_B] \\ &= g \circ f(g'(b)) && [\text{by associativity of compositions}] \\ &= \mathrm{id}_A(g'(b)) && [\text{since } g \circ f = \mathrm{id}_A] \\ &= g'(b) \ . \end{aligned}$$

Since b was arbitrary, we have shown that $g = g'$. □

Exercises

1. Let A, B and C be sets. Prove the *Distributive law*

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

via element-chasing (page 3):

2. Let A, B and C be sets. Prove *DeMorgan's laws* via element-chasing (page 3):
   (a) $A - (B \cup C) = (A - B) \cap (A - C)$.
   (b) $A - (B \cap C) = (A - B) \cup (A - C)$.
3. Let A, B, C and D be sets. Prove the following via element-chasing (page 3):
   (a) $(A \times B) \cup (C \times D) \subset (A \cup C) \times (B \cup D)$.
   (b) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.
   (c) $A \times (B - C) = (A \times B) - (A \times C)$.
   (d) $(A - C) \times (B - D) \subset (A \times B) - (C \times D)$.

   For parts (a) and (d), explain why equality does not hold; to do this, come up with a counter-example for the reverse containment.
4. Prove that a function $f : A \to B$ is invertible if and only if it is bijective.
5. Prove the following via element-chasing (page 3): Let $f : A \to B$ be a function. Let U and V be subsets of B. Then
   (a) $U \subset V \Rightarrow f^{-1}(U) \subset f^{-1}(V)$.
   (b) $f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$.
   (c) $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$.
   (d) $f(f^{-1}(U)) \subset U$.
   (e) $X \subset A \Rightarrow X \subset f^{-1}(f(X))$.
   (f) $[f(f^{-1}(U)) = U$ for every $U \subset B] \Rightarrow f$ is surjective.
   (g) $[f^{-1}(f(X)) = X$ for every $X \subset A] \Rightarrow f$ is injective.

References

[Mun75] James R. Munkres. *Topology: a first course.* Prentice-Hall Inc., Englewood Cliffs, N.J., 1975.