MAT 145 : Quiz Solutions

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Quiz 1 Solutions

1. Let the set $X = \{a, b, c, d\}$ be given the topology $T = \{\emptyset, X, \{c\}, \{c, d\}, \{a, b, c\}\}$. Let $S$ be the subset $S = \{a, c, d\} \subset X$.

(a) List the elements of the subspace topology $T_S$ on $S$.

Solution: By definition of $T_S$, we have $T_S = \{O \cap S : O \in T\}$. Therefore $T_S = \{\emptyset, S, \{c\}, \{d\}, \{c, d\}, \{a, c\}\}$. 

(b) With respect to the topologies $T$ on $X$ and $T_S$ on $S$, determine whether or not the function $f: S \rightarrow X$, $f(a) = a$, $f(c) = d$, $f(d) = c$ is continuous. Justify your answer.

Solution: The function $f$ is not continuous: for $f$ to be continuous, the pre-image of any open set of $X$ has to be an open set of $S$; in other words, $f^{-1}(O) \in T_S$ whenever $O \in T$. By considering the pre-image of each point in $\{a, b, c\}$, we see that $f^{-1}(\{a, b, c\}) = \{a, d\}$. Since $\{a, b, c\} \in T$ while $f^{-1}(\{a, b, c\}) = \{a, d\} \notin T_S$, $f$ cannot be continuous.

Quiz 2 Solutions

1. Let $X$ be a space. For any subset $A \subset X$, prove that

\[ \partial A = \emptyset \iff A \text{ is both open and closed in } X. \]
**Elementary Solution:** Recall the definition of $\partial A$:

$$\partial A = \{x \in X : \text{every neighborhood of } x \text{ intersects } A \text{ and } X - A\}.$$ 

Also recall that $A$ is closed if and only if $X - A$ is open. Suppose that $\partial A = \emptyset$. Given $x \in X$, there exists a neighborhood $O_A$ of $x$ that does not intersect $X - A$ (hence $x \in O_A \subset A$), or there exists a neighborhood $O_{X-A}$ of $x$ that does not intersect $A$ (hence $x \in O_{X-A} \subset X - A$). In particular, for every $x \in A$, there exists a neighborhood $O_A$ of $x$ such that $x \in O_A \subset A$, so $A$ is open; similarly, for every $x \in X - A$, there exists a neighborhood $O_{X-A}$ of $x$ such that $x \in O_{X-A} \subset X - A$, so $X - A$ is open. This establishes that $\partial A = \emptyset$ implies that $A$ is open and closed.

Now suppose that $A$ is open and closed; hence $A$ and $X - A$ are open. Let $x \in X$. We show that $x \notin \partial A$. If $x \in A$, then $A$ itself is a neighborhood $x$ that does not intersect $X - A$ (because $A$ is open). If $x \in X - A$, then $X - A$ itself is a neighborhood of $x$ that does not intersect $A$ (because $X - A$ is open). In either case ($x \in A$ or $x \in X - A$), we have that $x \notin \partial A$. This establishes that if $A$ is open and closed, then $\partial A = \emptyset$.

**Nonelementary Solution:** From class and Hatcher’s notes, we may use the facts

1. $\text{int}(A) \cup \partial A = \bar{A}$.
2. $\text{int}(A) \subset A \subset \bar{A}$.
3. $A$ is open if and only if $\text{int}(A) = A$; $A$ is closed if and only if $A = \bar{A}$.

By (1), we immediately have $\partial A = \emptyset$ if and only if $\text{int}(A) = \bar{A}$. By combining this with (2), we see that $\partial A = \emptyset$ if and only if $\text{int}(A) = A = \bar{A}$. Therefore, by (3), $\partial A = \emptyset$ if and only if $A$ is open ($\text{int}(A) = A$) and closed ($A = \bar{A}$).

2. Let $X$ be a Hausdorff space, and let $A$ be a subspace of $X$. Prove that $A$ is a Hausdorff space.

**Solution:** Let $X$ be a Hausdorff space, and let $A$ be a subspace of $X$. For clarity, we let $T_X$ denote the given topology on $X$, and let $T_A$ be the induced subspace topology on $A$. Let $x_1$ and $x_2$ be distinct points in $A$; we will show that there exists disjoint neighborhoods (from $T_A$) of $x_1$ and $x_2$. Since $X$ is Hausdorff, there exists a neighborhood $O_1' \in T_X$ of $x_1$ and there exists a neighborhood $O_2' \in T_X$ of $x_2$ such that $O_1' \cap O_2' = \emptyset$. Define $O_1 = O_1' \cap A \in T_A$, and define $O_2 = O_2' \cap A \in T_A$. By definition of $T_A$, we see that $O_1$ is a neighborhood of $x_1$ (open in $A$) and $O_2$ is a neighborhood of $x_2$ (open in
A). We also see that

\[ O_1 \cap O_2 = (O'_1 \cap A) \cap (O'_2 \cap A) = (O'_1 \cap O'_2) \cap A = \emptyset, \]

since \( O'_1 \cap O'_2 = \emptyset \). This establishes that the subspace \( A \) is a Hausdorff space.

3. Let the set \( X = \{a, b, c, d\} \) be given the topology

\[ T = \{\emptyset, X, \{c\}, \{a, b, c\}\}. \]

(a) Prove or disprove: \( X \) is Hausdorff.

**Solution:** We prove that \( X \) is not Hausdorff. Observe that the only neighborhood of \( d \) is the whole space \( X \) because \( d \notin \{c\} \) and \( d \notin \{a, b, c\} \). Also observe that the only neighborhoods of \( b \) are \( X \) and \( \{a, b, c\} \). So \( d \) and \( b \) do not possess disjoint neighborhoods. Therefore \( X \) is not Hausdorff.

(b) Prove directly that \( X \) is connected.

**Solution:** Suppose that \( X = A \cup B \) were a separation of \( X \) (i.e. \( A \) and \( B \) are disjoint nonempty open sets whose union is \( X \)); we derive a contradiction. Observe that the only neighborhood of \( d \) is the whole space \( X \). Since \( d \) lies in either \( A \) or \( B \), we deduce that either \( A = X \) or \( B = X \); so, either \( B = \emptyset \) or \( A = \emptyset \). This contradicts that \( X = A \cup B \) is a separation.

(c) Show that the subspace \( S = \{a, b\} \) is path connected by explicitly defining a path between \( a \) and \( b \).

**Solution:** Define a function \( f : [0, 1] \to S \) by \( f(t) = a \) for all \( 0 \leq t \leq 1/2 \), and \( f(t) = b \) for all \( 1/2 < t \leq 1 \). Note that the subspace topology on \( S \) is the indiscrete topology. It is easy to see that \( f \) is continuous (indeed, \( f^{-1}(\emptyset) = \emptyset \) and \( f^{-1}(S) = [0, 1] \) are open in \([0, 1]\)). Therefore, \( f \) is a path in \( S \) from \( a \) to \( b \).

(d) For each pair or points in \( X \), explicitly define a path between these points. Deduce that \( X \) is actually path connected.

**Solution:** *This is similar to the the construction in part (c), but some care has to be taken when constructing paths involving the point \( c \):*

To construct a path from \( a \) to \( c \), define a function \( f : [0, 1] \to X \) by \( f(t) = a \) for all \( 0 \leq t \leq 1/2 \), and \( f(t) = c \) for all \( 1/2 < t \leq 1 \). All the relevant pre-images \( f^{-1}(\emptyset) = \emptyset, f^{-1}(X) = [0, 1], f^{-1}(\{c\}) = (1/2, 1], \) and \( f^{-1}(\{a, b, c\}) = [0, 1] \) are all open in \([0, 1]\). So \( f \) is a (continuous) path.
To construct a path from $b$ to $c$, define a function $f : [0, 1] \to X$ by $f(t) = b$ for all $0 \leq t \leq 1/2$, and $f(t) = c$ for all $1/2 < t \leq 1$. All the relevant pre-images $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(X) = [0, 1]$, $f^{-1}(\{x\}) = (1/2, 1]$, and $f^{-1}(\{a, b, c\}) = [0, 1]$ are all open in $[0, 1]$. So $f$ is a (continuous) path.

To construct a path from $d$ to $c$, define a function $f : [0, 1] \to X$ by $f(t) = d$ for all $0 \leq t \leq 1/2$, and $f(t) = c$ for all $1/2 < t \leq 1$. All the relevant pre-images $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(X) = [0, 1]$, $f^{-1}(\{x\}) = (1/2, 1]$, and $f^{-1}(\{a, b, c\}) = (1/2, 1]$ are all open in $[0, 1]$. So $f$ is a (continuous) path.

To construct a path from $d$ to $a$, define a function $f : [0, 1] \to X$ by $f(t) = d$ for all $0 \leq t \leq 1/2$, and $f(t) = a$ for all $1/2 < t \leq 1$. All the relevant pre-images $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(X) = [0, 1]$, $f^{-1}(\{x\}) = \emptyset$, and $f^{-1}(\{a, b, c\}) = [0, 1]$ are all open in $[0, 1]$. So $f$ is a (continuous) path.

A path from $a$ to $b$ was already constructed in part (c). We can use the same formula for $f$. All the relevant pre-images $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(X) = [0, 1]$, $f^{-1}(\{x\}) = \emptyset$, and $f^{-1}(\{a, b, c\}) = [0, 1]$ are all open in $[0, 1]$. So $f$ is a (continuous) path.

Since we were able to construct all of the required paths in $X$, we deduce that $X$ is path connected.

**Quiz 3 Solutions**

1. Give a self-contained proof of the following: Let $X$ be a path connected space, and let $Y$ be a space. Suppose that $f : X \to Y$ is a surjective continuous function. Show that $Y$ is path connected.

**Solution:** Let $y_1, y_2 \in Y$; we show that there exists a path in $Y$ from $y_1$ to $y_2$. Since $f$ is surjective, there exists $x_1, x_2 \in X$ for which $f(x_1) = y_1$ and $f(x_2) = y_2$. Since $X$ is path connected, there exists a continuous function $g : [0, 1] \to X$ for which $g(0) = x_1$ and $g(1) = x_2$. Since $f$ and $g$ are continuous, the composition $(f \circ g) : [0, 1] \to Y$ is continuous. Furthermore, $(f \circ g)(0) = f(x_1) = y_1$ and $(f \circ g)(1) = f(x_2) = y_2$. Therefore, a path in $Y$ from $y_1$ to $y_2$ exists. This establishes that $Y$ is path connected.

2. Let $X$ be a compact Hausdorff space, and let $A$ be a closed subset of $X$. Suppose that $y \in X - A$. Prove that there exist open sets $V$ and $V'$ in $X$ such that $y \in V$, $A \subset V'$, and $V \cap V' = \emptyset$. 

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Solution: Given $a \in A$, there exists disjoint neighborhoods $V_a$ and $V'_a$ of $y$ and $a$ respectively in $X$, since $X$ is Hausdorff. We see that $\{V'_a \cap A\}_{a \in A}$ forms an open covering of $A$. Since $A$ is closed in the compact space $X$, we have that $A$ is compact. So there exists a finite subcovering $\{V'_a \cap A, \ldots, V''_a \cap A\}$ of $A$. Set $V = \bigcap_{i=1}^n V_{a_i}$ and $V' = \bigcup_{i=1}^n V'_{a_i}$. Since $y \in V$, for each $i = 1, \ldots, n$, we see that $y \in V_i$; since $V_{a_i} \cap V'_{a_i} = \emptyset$ for each $i = 1, \ldots, n$, we see that $V \cap V' = \emptyset$. Since $V$ is the finite intersection of open sets in $X$, we see that $V$ is open in $X$. Since $V'$ is the union of open sets of $X$, we see that $V'$ is open in $X$. Finally, we see that $A \subset \bigcup_{i=1}^n V'_{a_i} = V'$. Therefore, there exist open sets $V$ and $V'$ in $X$ such that $y \in V$, $A \subset V'$, and $V \cap V' = \emptyset$. 