# More Knots in Knots: a study of classical knot diagrams

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ABSTRACT

The structure of classical minimal prime knot presentations suggests that there are often, perhaps always, subsegments that present either the trefoil or the figure-eight knot. A comprehensive study of the subknots of the minimal prime knot presentations through 15 crossings shows that this is always the case for these knot presentations. Among this set of $313,258$ prime knot presentation, there are only 547, or 0.17%, that do not contain a trefoil subknot. Thus, 99.83% of minimal prime knot presentations through 15 crossings contain trefoil subknots. We identify several infinite minimal alternating prime knot families that do not contain trefoil subknots but always contain figure-eight knots. We discuss the statistics of subknots of prime knots and, using knot presentation fingerprints, illustrate the complex character of the subknots of these classic minimal prime knot presentations. We conclude with a discussion the conjectures and open questions that have grown out of our research.

**Keywords:** knot presentation, minimal crossing, prime knots, subknots

## 1. Introduction

In a May 2009 lecture during the Conference on Knot Theory and its Applications to Physics and Biology held at the Abdus Salam International Centre for Theoretical Physics Advanced School in Trieste, Italy, two conjectures concerning the fine structure of prime knots, understood as either minimal crossing prime knot diagrams or ideal representations of the prime knot type, were proposed. They are:

**First Conjecture** *The probability that a minimal prime knot diagram contains a trefoil segment goes to one as the crossing number goes to infinity.*

**Second Conjecture** *The probability that a minimal prime knot diagram contains a trefoil slipknot segment goes to one as the crossing number goes to infinity.*

In this report, we describe our efforts in support of the first conjecture and to confirm, at least for minimal prime knot diagrams through 15 crossings, the following conjecture:

**Third Conjecture** *Every minimal prime knot diagram contains a trefoil or figure-eight knotted segment.*

In 2006, Millett had noticed that many of the presentations of prime knots in standard tables, e.g. Rolfsen [10], contained subsegments that were intrinsically trefoil or figure-eight knotted arcs. After a preliminary visual analysis, the project of a systematic analysis of the known prime knots was undertaken in collaboration with Joseph Migler, who wrote a computer code that identified all those containing trefoil knots. Supplemented with a case by case visual analysis of those remaining, it seemed that every prime knot contained either trefoil or figure-eight knots or both. This data lead to the conjectures proposed in the 2009 lecture. Slavik Jablan and Ken Millett explored these conjectures from which a more precise interpretation of the conjectures was formulated as reported in [6]. In this report, the preliminary results of a computer implementation of the refined analysis of all prime knots through 16 crossings was reported. In this report, we give a brief historical account of these conjectures ending with their present formulation. We describe the implementation of a new exhaustive analysis of the subknots of prime knot diagrams through 15 crossings in light of the conjectures, give the data for all prime knots through 10 crossings and describe the character of the data for the prime knots from 11 through 15 crossings, and describe several infinite families of minimal crossing alternating prime knots that do not contain trefoil subknots but always contain figure-eight subknots. Access to the data set is provided at the website http://www.math.ucsb.edu/ millett/KnotsinKnots. We conclude with a discussion of the conclusions drawn from the analysis of the data and their implications for the conjectures and further research.

### 1.1. *Historical Perspectives*

Seeking a mathematically robust method to identify knotting in open arcs, for example protein structures, in contrast to the classical mathematical theory of knotting that is restricted to circular arcs, Dobay, Millett, and Stasiak [7] developed and employed a new method that could be applied to open molecular chains such as proteins or other macromolecular structures modeled by open arcs. The method uses a statistical analysis of the knotting in a collection of closures of the arcs by connecting its ends to points on a very large sphere containing the open chain, approximating the "sphere at infinity". With this new method, one is able to assess the influence of knotting in an open chain on its radius of gyration and determine how it scales with increasing length. Millett and Sheldon [8] reassessed the presence of knots in proteins, Millett [5] measured the average size of knots and slipknots in random walks, and, more recently, Sulkowska et. al. [11] undertook a systematic analysis of known protein structures and the biological character of those containing knots. In 2006, Millett noticed that many of the presentations of prime knots in standard tables, e.g. Rolfsen [10], contained subsegments that were intrinsically trefoil or figure-eight knotted arcs. After a preliminary visual analysis, a systematic analysis of the known prime knots was undertaken by Joseph Migler and Millett using a computer code that identified all those containing trefoil knots. Supplemented by a careful visual analysis of the remaining cases, it seemed that all prime knots contained either a trefoil or a figure-eight knot or both. The resulting data lead to the conjectures proposed in the 2009 lecture and the research reported in [6] and this note.

### 1.2. *Subknots of a Knot*

The fundamental challenge is to determine "How can one identify a knot supported on a subsegment of a minimal crossing prime knot diagram?" To describe our strategy, we consider the case of the $K11a135$ knot is shown in Figure 1. Imagine that the entire figure lies in the plane except for small "indentations" at the crossings where the lower strand dips below the plane. For example, we take the short segment of $K11a135$ indicated by the starting green circle, the terminating red circle, and containing the orientation arrow as shown in the center. One might expect that almost all of the closures would result in a trivial knot due to the essentially planar character of the segment and its evidently unknotted character. In fact, as the initial and terminal points of the segment lie in the same complementary region of the complement of the projection of the segment, shown on the right of Figure 1, center, almost all closures result in the unknot.

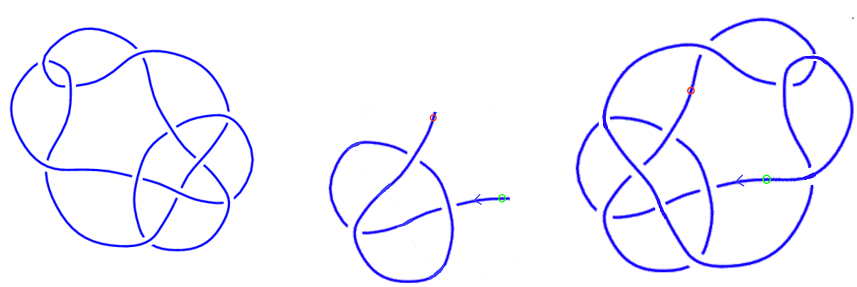

Fig. 1. Presentation of $K11a135$ on the left. In the center, a short initial segment of $K11a135$ oriented in the direction of the arrow, starting at the green circle, and ending at the red circle is shown. The remaining portion of the $K11a135$ diagram is modified to represent a strictly monotonically rising complementary segment passing over the short segment until just before arriving at the green circle where is descends rapidly to complete the circuit. The short segment, with the modified complementary segment is shown on the left.

Jablan and Millett explored several possible definitions that might better capture the intent of the conjectures and to, eventually, facilitate a rigorous proof of the conjecture(s) without employing the statistics of the closures to 'the sphere at infinity.' We focused on crossing changes modifying the minimal presentation of the prime knot consistent with the focus on the structure of subsegments and proposed the following definiton [6]:

**Definition** *A knot, K*, will be called a* ***subknot*** *of a knot K if it can be obtained from a minimal diagram of K by crossing changes that preserve those within a segment of the diagram and change those outside this segment so that the complementary segment is strictly ascending and lies over the subsegment.*

Observe that this captures the structure in the $K11a135$ subknot case, a trivial knot, described above. The result of the crossing changes for the complementary subsegment are realized in the right in Figure 1 giving a classical trivial knot.

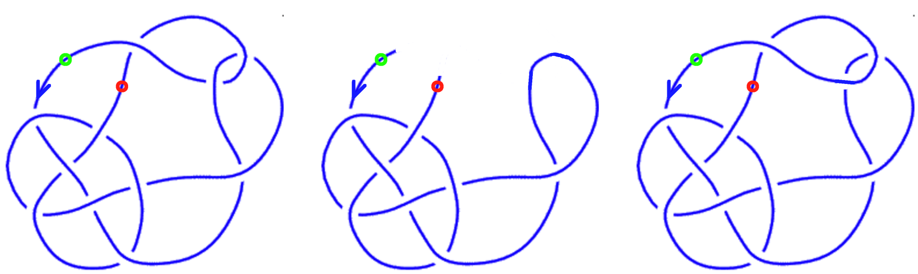

Fig. 2. Presentation of $K11a135$, a long segment starting at the green circle in the direction of the arrow until the red circle giving a $7_7$ subknot, the crossings of the complementary segment removed and, the equivalent result with the complementary segment crossings changed as required.

A more complex case is shown in Figure 2, where we consider a larger subsegment giving the $7_7$ knot in $K11a135$ as shown in the left. Removing the complementary segment leaves an arc whose majority closure is a $7_7$ knot. Here, again, we change the crossings in the complementary segment to achieve a strictly ascending structure lying over the selected subknot and show the result on the right in Figure 2. If one further shortens this subsegment as shown in Figure 3, the result is a figure-eight knot, $4_1$. As for the previous $7_7$ case, we also show the configuration with the complementary segment removed and with the crossings of the complementary segment changed to achieve a strictly ascending segment lying over the selected subsegment in Figure 3.

The knot fingerprint [9], shown in Figure 4, is adapted to the case of subsegments of a minimal crossing prime knot presentation and shows the entire specturm of subknots of the given presentation. The circular array is indexed by the length of the segment of crossings, each crossing being counted twice, so that each under crossing in made an over crossing exactly once and each over crossing remains unchanged. The knot types of the segments are coded by color according to the color code

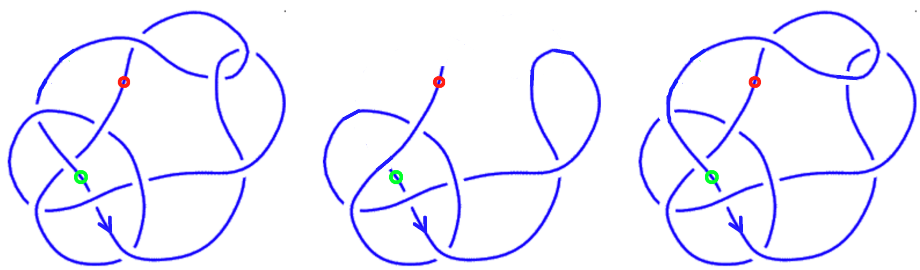

Fig. 3. Presentation of $K11a135$, a long segment starting at the green circle in the direction of the arrow until the red circle giving a $4_1$ subknot, the crossings of the comple mentary segment removed and, the equivalent result with the complementary segment crossings changed as required.

provided to the left of the circular array. In the array one can observe the presence of the blue figure-eight knot, $4_1$, contained within the pink $7_7$ by following the ray from the red unknot center at about 3 o'clock toward the boundary. The $K11a135$ knot is an example of an alternating prime minimal crossing knot that does not contain a trefoil knot. The $K11n123$ is a non-alternating prime minimal crossing knot that does not contain a trefoil but it too contains a figure-eight knot. Note that by looking at the bounday of the knotting fingerprint one can observe that the knot diagram contains two single crossing changes that unknot it. These are the only two 11 crossing minimal crossing prime knot diagrams that do not contain trefoils.

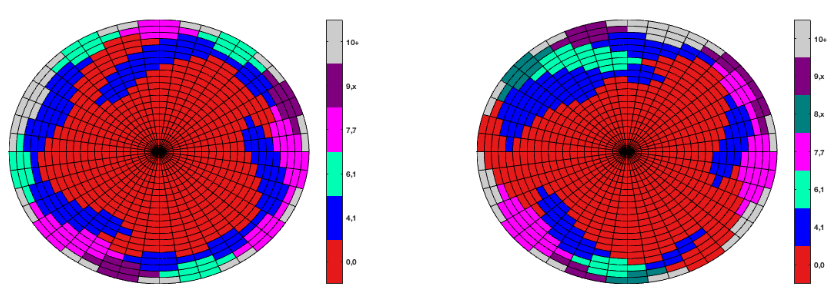


Fig. 4. Both $11a_{135}$ and $11n_{123}$ minimal diagram knot presentations contain figure-eight knots but do not contain trefoils.

## 2. Subknot Data Generation

Given a minimal crossing diagram of a prime knot, we must systematically enumerate all segments defined by cutting the implied circle at points between crossings and identify the knot type to be associated to the segment. Even though the number of subsegments of each diagram grows quadratically with the number of crossings, the number of minimal crossing diagrams grows exponentially making the efficiency of the method of great importance.

The algorithm for subknot generation employs 5 steps:

(1) The systematic idenfification of all subsegments of a give knot presentation.
(2) Identifying and preserving crossings within the subsegment.
(3) Lifting the crossings of complimentary segment above those of the subsegment.
(4) Changing the appropriate crossings in the complimentary subsegment as as to be strictly ascending.
(5) Simplifying the resulting knot presentation and identifying the knot type of the subsegment.

Several different programs are used to generate all possible subknot data from a given knot. First, a Matlab code is used to generate a list of altered code from input of the prime knot's minimal diagram Dowker-Thistlethwaite (DT) code $I$, [3]. The program does this in several steps. The input DT code is separated into two different sequences of information: crossing data $D$ and sequential data $S$. Given an $n$-crossing knot, the crossing data is presented in the form of a $2 \times n$ matrix, with each column representing the signed crossings as given by the DT code. Sequential data is represented in a $1 \times 2n$ vector recording the expanded sequential path indicated by the Dowker-Thistlethwaite algorithm from crossing point 1 until crossing point $2n$. The knot $5_1$, see Figure 5, with DT code

$$I = \begin{bmatrix} 6 \; 8 \; 10 \; 2 \; 4 \end{bmatrix}$$

has crossing data

$$D = \begin{bmatrix} -1 & -3 & -5 & -7 & -9 \\ 6 & 8 & 10 & 2 & 4 \end{bmatrix}$$

and sequential data

$$S = \begin{bmatrix} -1 \; 2 \; -3 \; 4 \; -5 \; 6 \; -7 \; 8 \; -9 \; 10 \end{bmatrix}$$

Subsegments are defined using the sequential path. The algorithm sequentially considers all possible starting points $p \in S$ and all possible segment lengths $l$ such that $l < 2n$. The subsegment $U_{p,l}$ is defined to start immediately before crossing point $p$ and end immediately after crossing point $(p + l - 1) mod(2n)$. We represent this by extracting a vector of length $1 \times (p + l - 1)$ directly from our sequence $S$. In

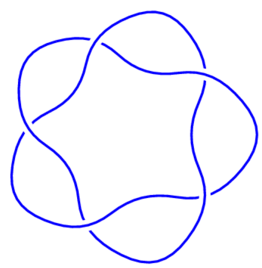

Fig. 5. Knot $5_1$

our example, the subsegment that starts immediately before crossing point $p = 7$ of length $l = 6$ is defined to be

$$U_{7,6} = \left[-7\ 8\ -9\ 10\ -1\ 2\right]$$

Note that, from our definition, the number of possible subsegments of an $n$ crossing knot is $2n(2n-1)$.

Once a subsegment has been defined, the program, using crossing data $D$, applies steps $2-4$ to the crossings within the segment, the crossings between the segment and its compliment, and then, the crossings within the compliment. Crossings within the segment are preserved in the output DT code. In $U_{8,7}$ we see crossings $(-3, 8)$, and $(-9, 4)$, and thus preserve the sign of 8 and 4 in our output. Crossings of the segment with the compliment are then identified and analyzed to determine which section passes above. If the segment passed above the compliment, the signs of the crossing number are switched to reflect lifting the compliment above the segment. In our example, we have crossings $(-1, 6)$, $(-5, 10)$ and $(-7, 2)$. -1 is in our segment and passes above 6, thus we switch the signs to produce output crossing $(1, -6)$. The remaining crossings have the segment passing beneath the compliment, and are therefore preserved in the output. The compliment is then scanned, starting from $(p+l-1)mod(2n)$ to $(p-1)mod(2n)$. Any crossings within the compliment such that the later section passed beneath the earlier section has its signs switched to reflect strict ascending structure of the compliment. In our example, the compliment does not have self-crossings; however, in $U_{7,4}$ we observe crossing $(-1, 6)$ in our output, and record $(1, -6)$ to reflect strict ascension.

We have produced a DT code that is not in simplest form. In our example with segment $U_{8,7}$, the output is

$$O_{8,7} = \left[-6\ 8\ 10\ 2\ 4\right]$$

A quick visual analysis confirms that this is a trefoil. For larger knots, this is not so easy to see. Try, for instance, to simplify the subknot produced from segment $U_{4,19}$ of knot $11a_{135}$. While it is possible to do visually (it is $7_7$), we utilized a program, *unraveller* provided by Morwen Thistlethwaite, to simplify the initial raw subknot DT code. The program takes DT code as input, and outputs DT code in a simpler form, allowing us to more easily identify its associated knot type. This code easily simplifies both of our examples, producing the DT code for $3_1$ and $7_7$ respectively. Using these tools, we are able to generated all possible subknots of a given knot and identify their topological type. The results are then organized and analyzed using a Perl program. These base programs are applied to the lists of presentations of minimal crossing prime knots and to generate our final data. This data, through 15 crossings, can be found at http://www.math.ucsb.edu/millett/KnotsinKnots.

## 3. Subknots of Minimal Prime Knot Diagrams

Following an initial visual analysis of the first minimal crossing prime knot diagrams, through eight crossings, three phenomena are apparent: (1) not all diagrams contain trefoils, e.g. the figure-eight diagram does not contain a trefoil; (2) a large number of diagrams do contain trefoils; and (3) every diagram contained either a trefoil (all but $4_1$ and $6_1$), or a figure-eight subknot (confirmed in $4_1$ and $6_1$). Joseph Migler then wrote a program to search minimal crossing prime knot diagrams for the presence of trefoil knots [12,3]. Those that did not contain trefoils were visually analyzed to determine if they contained figure-eight knots. Encouraged by this primitive analysis, we undertook the systematic study of the subknots of the standard minimal crossing prime knot presentations.

### 3.1. *Computer Analysis of Knot Types of Subknots*

Table 1 reports the percentage of distinct trefoil segments and the percentage of figure-eight knotted segments among all subknot segments of prime knots through seven crossings. For example, there are 30 segments in the 3 crossing trefoil. In order to have a trefoil subknot, according to our definition, the complementary segment must be exactly a single over crossing. As a consequence, the trefoil contains exactly 3 trefoil, i.e. $3_1$, subknots or 10% and no figure-eight, i.e. $4_1$, subknots as indicated in the first row of the table. In the same way, of the 56 subknots of the 4 crossing figure-eight knot, there are exactly four $4_1$ subknots and no $3_1$ subknots as is reported in the second row of the table. Note that the fourth column reports the number of distinct knot types observed among all the subknots. Two types always occur, the unknot and the knot type of the diagram. Thus, for example, for the $5_1$ knot, we observe $0_1$, $3_1$ and, $5_1$ or three distinct knot types as is indicated in the third row of the table.

Table 2 reports the analysis of the eight crossing prime knots. Among the 21 eight crossing prime knots, there are 3 that do not contain $3_1$'s: $8_1, 8_3, \&\ 8_{12}$. As earlier, each of these does, however, contain $4_1$ subknots.

All nine crossing prime knots, Table 3, contain $3_1$'s.

Tables 4 and 5, for ten crossing prime knots, show that 96% contained trefoil

Table 1: Analysis of Subknots in Minimal 3 to 7 Crossing Prime Knots

| Knot | percentage $3_1$ | percentage $4_1$ | Unique Subknots |
|---|---|---|---|
| $3_1$ | 10 | 0 | 2 |
| $4_1$ | 0 | 7.143 | 2 |
| $5_1$ | 22.222 | 0 | 3 |
| $5_2$ | 13.333 | 0 | 3 |
| $6_1$ | 0 | 12.121 | 3 |
| $6_2$ | 9.09 | 9.09 | 4 |
| $6_3$ | 26.667 | 0 | 3 |
| $7_1$ | 15.38 | 0 | 4 |
| $7_2$ | 6.593 | 0 | 4 |
| $7_3$ | 10.981 | 0 | 5 |
| $7_4$ | 8.791 | 0 | 4 |
| $7_5$ | 15.38 | 0 | 5 |
| $7_6$ | 10.981 | 6.593 | 5 |
| $7_7$ | 2.198 | 13.28 | 4 |

knots. The 7 minimal 10 crossing prime knots without trefoils are $10_1$, $10_3$, $10_{13}$, $10_{35}$, $10_{45}$, $10_{58}$, and $10_{123}$.

For the 552 eleven crossing knots, 99.6% contain trefoils. The only two knots without trefoils are $11a_{135}$ and $11n_{123}$. The contain figure-eight subknots.

For the $2,176$ twelve crossing knots, 98.6% contain trefoils. There are 31 that do not contain trefoils. They contain figure-eight subknots.

For the $9,988$ thirteen crossing knots, 99.7% contain trefiols. There are 29 that do not contain trefoils. They contain figure-eight subknots.

For the $46,972$ fourteen crossing knots, 99.77% contain trefoils. Thus, 0.23% or 109 do not contain trefoils. They contain figure-eight knots.

For the $253,263$ fifteen crossing knots, 99.86% contain trefoils. Thus, 0.14% or 366 do not contain trefoils. They contain figure-eight knots.

## 4. A Ubiquitous Family: Trefoils, Figure-Eights and, Others?

So far, every minimal diagram of a prime knot has contained either a trefoil or a figure-eight knot. Perhaps, if our conjecture is not true, there may still be a finite collection of prime knot types with the property that every minimal diagram contains one of these. We considered possible prime knot types that might contain neither a trefoil nor a figure-eight knot but have failed to identify a counter-example to the conjecture. Interesting possible cases have included the Conway knots, $8^* = 8_{18}$, $9^* = 9_{40}$, and $10^* = 10_{123}$, Figure 8. The first case, $8^*$, has been discussed in [6] and does contain trefoils. In Table 3 we see that $9_{40}$ does contain trefoils but that, in Table 5, $10_{123}$ does not contain trefoils. Both of contain figure-eight knots.

| Table 2: Analysis of Subknots in Minimal 8 Crossing Prime Knots | | | |
|---|---|---|---|
| Knot | percentage $3_1$ | percentage $4_1$ | unique subknots |
| $8_1$ | 0 | 0.066 | 4 |
| $8_2$ | 0.1 | 0.05 | 6 |
| $8_3$ | 0 | 0.066 | 4 |
| $8_4$ | 0.05 | 0.066 | 6 |
| $8_5$ | 0.166 | 0.033 | 6 |
| $8_6$ | 0.1 | 0.066 | 6 |
| $8_7$ | 0.15 | 0 | 5 |
| $8_8$ | 0.183 | 0 | 5 |
| $8_9$ | 0.1 | 0.033 | 5 |
| $8_{10}$ | 0.216 | 0 | 7 |
| $8_{11}$ | 0.033 | 0.05 | 7 |
| $8_{12}$ | 0 | 0.166 | 4 |
| $8_{13}$ | 0.083 | 0.016 | 6 |
| $8_{14}$ | 0.066 | 0.1 | 6 |
| $8_{15}$ | 0.2 | 0 | 6 |
| $8_{16}$ | 0.15 | 0.016 | 6 |
| $8_{17}$ | 0.116 | 0.033 | 5 |
| $8_{18}$ | 0.133 | 0 | 3 |
| $8_{19}$ | 0.225 | 0 | 7 |
| $8_{20}$ | 0.158 | 0 | 7 |
| $8_{21}$ | 0.166 | 0.025 | 6 |

Thus, so far, all the evidence suggests that the conjecture may be true.

### 4.1. *Knots Without Trefoil Subknots*

To further test the conjecture, we analyze the known minimal crossing prime knot presentations that do not contain trefoils with the twin goal of proving the existence of infinite families of knots without trefoil subknots and to determine whether or not each of them contains figure-eight knots. The first class of knots to consider are the rational knots, i.e. those given by the Conway notation $[n_1\ n_2 \dots n_k]$ [2]. The first class consists of $4_1$=[ 2 2 ], $8_{12}$=[ 2 2 2 2 ], $12a_{477}$=[ 2 2 2 2 2 2 ], ..., $6_1$=[ 4 2 ], $10_{13}$=[ 4 2 2 2 ], ..., $8_1$=[ 6 2 ], $12a_{691}$=[ 6 2 2 2 ], ..., $10_1$=[ 8 2 ], $14a$??=[ 8 2 2 2 ], ..., $12a_{803}$=[ 10 2 ], .... This class also includes $8_3$ = [ 4 4], $10_3$ = [ 6 4 ], $10_{13}$ = [ 4 2 2 2], $10_{35}$ = [2422], $12a_{1166}$ = [ 8 4 ], $12a_{482}$ = [ 4 4 2 2 ], $10_{35}$ = [ 2 4 2 2 ], $12a_{197}$=[ 2 6 2 2 ], $12a_{471}$=[2442],$12a_{690}$=[4242],$12a_{1127}$=[4224],$12a_{1287}$=[66].

**Theorem 4.1.** *Knots with a Conway notation of the form* [ $k_1\ k_2 \dots k_n$ ] *consisting of sequence of* $n \geq 2$ *positive even numbers* $k_1, k_2, \dots, k_n$ *do not contain trefoil subknots but always contain* $4_1$ = [ 2 2 ] *knots.*

Table 3: Analysis of Subknots in Minimal 9 Crossing Prime Knots

| Knot | perc. $3_1$ | perc. $4_1$ | unique subknots | Knot | perc. $3_1$ | perc. $4_1$ | unique subknots |
|---|---|---|---|---|---|---|---|
| $9_1$ | 0.117 | 0 | 5 | $9_2$ | 0.039 | 0 | 5 |
| $9_3$ | 0.091 | 0 | 7 | $9_4$ | 0.065 | 0 | 7 |
| $9_5$ | 0.052 | 0 | 6 | $9_6$ | 0.117 | 0 | 7 |
| $9_7$ | 0.143 | 0 | 7 | $9_8$ | 0.130 | 0.052 | 7 |
| $9_9$ | 0.117 | 0 | 8 | $9_{10}$ | 0.065 | 0 | 7 |
| $9_{11}$ | 0.078 | 0.039 | 8 | $9_{12}$ | 0.065 | 0.026 | 8 |
| $9_{13}$ | 0.104 | 0 | 8 | $9_{14}$ | 0.026 | 0.065 | 7 |
| $9_{15}$ | 0.039 | 0.091 | 7 | $9_{16}$ | 0.196 | 0 | 8 |
| $9_{17}$ | 0.052 | 0.091 | 6 | $9_{18}$ | 0.065 | 0 | 8 |
| $9_{19}$ | 0.013 | 0.143 | 6 | $9_{20}$ | 0.130 | 0.013 | 8 |
| $9_{21}$ | 0.065 | 0.039 | 7 | $9_{22}$ | 0.098 | 0.091 | 10 |
| $9_{23}$ | 0.104 | 0 | 6 | $9_{24}$ | 0.156 | 0.039 | 9 |
| $9_{25}$ | 0.091 | 0.091 | 9 | $9_{26}$ | 0.091 | 0.052 | 8 |
| $9_{27}$ | 0.091 | 0.065 | 7 | $9_{28}$ | 0.183 | 0 | 6 |
| $9_{29}$ | 0.143 | 0.052 | 9 | $9_{30}$ | 0.111 | 0.078 | 10 |
| $9_{31}$ | 0.169 | 0 | 6 | $9_{32}$ | 0.078 | 0.052 | 9 |
| $9_{33}$ | 0.104 | 0.039 | 8 | $9_{34}$ | 0.052 | 0.078 | 6 |
| $9_{35}$ | 0.078 | 0 | 5 | $9_{36}$ | 0.124 | 0.065 | 11 |
| $9_{37}$ | 0.013 | 0.091 | 6 | $9_{38}$ | 0.169 | 0 | 8 |
| $9_{39}$ | 0.078 | 0.052 | 8 | $9_{40}$ | 0.039 | 0.078 | 5 |
| $9_{41}$ | 0.039 | 0.078 | 6 | $9_{42}$ | 0.078 | 0.088 | 10 |
| $9_{43}$ | 0.117 | 0.088 | 12 | $9_{44}$ | 0.084 | 0.055 | 11 |
| $9_{45}$ | 0.127 | 0.058 | 11 | $9_{46}$ | 0.045 | 0.062 | 8 |
| $9_{47}$ | 0.052 | 0.098 | 6 | $9_{48}$ | 0.140 | 0.013 | 7 |
| $9_{49}$ | 0.137 | 0 | 6 | | | | |

**Proof.** For these rational knots one may simply choose the complementary segment that reduces the second entry to 2, eliminates undercrossings in the remaining $k_i$, and, then, reduces $k_1$ to 2. An example implementing this observation for $12a471$ = [ 2 4 4 2 ] is shown in Figure 7. The fact that they do not contain trefoils follows from the observation that their subknots always have Conway notations with only positive even entries.

Note that the requirement that the $k_i$ are positive even is necessary because the trefoil knot has a Conway representation as [ 2 -2 ].

The 10 crossing knots without trefoil subknots are $10_{45} = [21111112]$, $10_{58} = [22, 22, 2]$, and $10_{123} = [10^*]$.

The 11 crossing knots without trefoils are $11a_{135} = [8 * 22]$ and $11n_{123} = [2.21, -2.2]$.

Table 4: Analysis of Subknots in Minimal 10 Crossing Prime Knots

| Knot | perc. $3_1$ | perc. $4_1$ | unique subknots | Knot | perc. $3_1$ | perc. $4_1$ | unique subknots |
|---|---|---|---|---|---|---|---|
| $10_1$ | 0 | 4.31 | 5 | $10_2$ | 8.421 | 3.257 | 8 |
| $10_3$ | 0 | 4.31 | 6 | $10_4$ | 3.158 | 4.31 | 8 |
| $10_5$ | 12.63 | 0 | 7 | $10_6$ | 7.368 | 4.31 | 9 |
| $10_7$ | 2.105 | 3.257 | 10 | $10_8$ | 6.316 | 4.31 | 9 |
| $10_9$ | 10.521 | 2.105 | 8 | $10_{10}$ | 5.263 | 2.105 | 9 |
| $10_{11}$ | 6.316 | 4.31 | 9 | $10_{12}$ | 13.68 | 0 | 9 |
| $10_{13}$ | 0 | 9.573 | 7 | $10_{14}$ | 7.368 | 6.415 | 10 |
| $10_{15}$ | 12.63 | 0 | 8 | $10_{16}$ | 2.105 | 4.31 | 10 |
| $10_{17}$ | 12.63 | 0 | 6 | $10_{18}$ | 5.263 | 3.257 | 9 |
| $10_{19}$ | 9.474 | 2.105 | 10 | $10_{20}$ | 11.571 | 4.31 | 8 |
| $10_{21}$ | 5.263 | 3.257 | 11 | $10_{22}$ | 9.474 | 4.31 | 8 |
| $10_{23}$ | 8.421 | 0 | 9 | $10_{24}$ | 3.158 | 4.31 | 10 |
| $10_{25}$ | 9.474 | 3.257 | 11 | $10_{26}$ | 5.263 | 4.31 | 10 |
| $10_{27}$ | 12.63 | 1.152 | 10 | $10_{28}$ | 10.521 | 1.152 | 9 |
| $10_{29}$ | 4.211 | 7.368 | 9 | $10_{30}$ | 3.158 | 6.415 | 10 |
| $10_{31}$ | 9.474 | 0 | 7 | $10_{32}$ | 11.571 | 1.152 | 9 |
| $10_{33}$ | 4.211 | 3.257 | 8 | $10_{34}$ | 16.84 | 0 | 7 |
| $10_{35}$ | 0 | 14.83 | 6 | $10_{36}$ | 4.211 | 11.67 | 8 |
| $10_{37}$ | 10.521 | 0 | 6 | $10_{38}$ | 6.316 | 4.31 | 9 |
| $10_{39}$ | 8.421 | 4.31 | 10 | $10_{40}$ | 11.571 | 0 | 8 |
| $10_{41}$ | 1.053 | 10.62 | 9 | $10_{42}$ | 5.263 | 6.415 | 9 |
| $10_{43}$ | 16.84 | 1.152 | 8 | $10_{44}$ | 7.368 | 7.368 | 9 |
| $10_{45}$ | 0 | 14.83 | 6 | $10_{46}$ | 15.781 | 2.105 | 10 |
| $10_{47}$ | 17.89 | 0 | 10 | $10_{48}$ | 17.89 | 0 | 11 |
| $10_{49}$ | 16.84 | 0 | 11 | $10_{50}$ | 11.05 | 2.105 | 13 |
| $10_{51}$ | 15.26 | 0 | 13 | $10_{52}$ | 13.151 | 1.152 | 13 |
| $10_{53}$ | 14.21 | 0 | 12 | $10_{54}$ | 17.89 | 0 | 11 |
| $10_{55}$ | 12.63 | 0 | 10 | $10_{56}$ | 13.68 | 2.105 | 12 |
| $10_{57}$ | 15.781 | 0 | 12 | $10_{58}$ | 0 | 17.46 | 7 |
| $10_{59}$ | 4.211 | 15.26 | 11 | $10_{60}$ | 1.053 | 14.83 | 9 |
| $10_{61}$ | 10.521 | 4.31 | 9 | $10_{62}$ | 15.781 | 0 | 11 |
| $10_{63}$ | 11.571 | 0 | 9 | $10_{64}$ | 13.68 | 1.152 | 9 |
| $10_{65}$ | 12.63 | 1.152 | 13 | $10_{66}$ | 15.781 | 0 | 10 |
| $10_{67}$ | 3.158 | 4.31 | 11 | $10_{68}$ | 8.421 | 1.152 | 9 |
| $10_{69}$ | 7.368 | 4.31 | 9 | $10_{70}$ | 6.842 | 11.67 | 12 |
| $10_{71}$ | 13.68 | 5.263 | 10 | $10_{72}$ | 10 | 8.421 | 13 |
| $10_{73}$ | 7.895 | 8.421 | 12 | $10_{74}$ | 4.211 | 4.31 | 10 |
| $10_{75}$ | 6.316 | 7.368 | 6 | $10_{76}$ | 18.941 | 3.257 | 9 |
| $10_{77}$ | 16.84 | 0 | 10 | $10_{78}$ | 13.68 | 3.257 | 8 |
| $10_{79}$ | 20 | 0 | 9 | $10_{80}$ | 20 | 0 | 11 |
| $10_{81}$ | 18.941 | 2.105 | 9 | $10_{82}$ | 11.571 | 2.105 | 10 |
| $10_{83}$ | 6.316 | 1.152 | 11 | $10_{84}$ | 18.941 | 0 | 9 |
| $10_{85}$ | 13.68 | 1.152 | 12 | $10_{86}$ | 5.263 | 3.257 | 9 |

Table 5: Analysis of Subknots in Minimal 10 Crossing Prime Knots

| Knot | perc. $3_1$ | perc. $4_1$ | unique subknots | Knot | perc. $3_1$ | perc. $4_1$ | unique subknots |
|---|---|---|---|---|---|---|---|
| $10_{87}$ | 14.731 | 2.105 | 11 | $10_{88}$ | 1.053 | 10.62 | 6 |
| $10_{89}$ | 2.105 | 9.573 | 9 | $10_{90}$ | 10.521 | 5.263 | 10 |
| $10_{91}$ | 17.89 | 0 | 10 | $10_{92}$ | 16.84 | 2.105 | 12 |
| $10_{93}$ | 14.731 | 4.31 | 12 | $10_{94}$ | 14.731 | 2.105 | 10 |
| $10_{95}$ | 17.89 | 0 | 12 | $10_{96}$ | 5.263 | 14.83 | 10 |
| $10_{97}$ | 8.421 | 8.421 | 11 | $10_{98}$ | 8.421 | 2.105 | 12 |
| $10_{99}$ | 18.941 | 0 | 9 | $10_{100}$ | 14.731 | 1.152 | 12 |
| $10_{101}$ | 12.63 | 0 | 11 | $10_{102}$ | 5.263 | 3.257 | 11 |
| $10_{103}$ | 13.68 | 1.152 | 10 | $10_{104}$ | 11.571 | 1.152 | 10 |
| $10_{105}$ | 5.263 | 7.368 | 10 | $10_{106}$ | 10.521 | 2.105 | 11 |
| $10_{107}$ | 10.521 | 3.257 | 11 | $10_{108}$ | 7.368 | 3.257 | 10 |
| $10_{109}$ | 17.89 | 0 | 9 | $10_{110}$ | 3.158 | 10.62 | 12 |
| $10_{111}$ | 14.731 | 4.31 | 11 | $10_{112}$ | 13.68 | 0 | 8 |
| $10_{113}$ | 9.474 | 5.263 | 9 | $10_{114}$ | 6.316 | 3.257 | 7 |
| $10_{115}$ | 7.368 | 2.105 | 8 | $10_{116}$ | 15.781 | 0 | 8 |
| $10_{117}$ | 11.571 | 0 | 11 | $10_{118}$ | 9.474 | 3.257 | 8 |
| $10_{119}$ | 4.211 | 8.421 | 12 | $10_{120}$ | 21.05 | 0 | 7 |
| $10_{121}$ | 10.521 | 2.105 | 9 | $10_{122}$ | 10.521 | 0 | 7 |
| $10_{123}$ | 0 | 10.62 | 4 | $10_{124}$ | 18.151 | 0 | 10 |
| $10_{125}$ | 18.68 | 0 | 11 | $10_{126}$ | 13.941 | 0 | 12 |
| $10_{127}$ | 17.361 | 1.678 | 11 | $10_{128}$ | 15 | 0 | 12 |
| $10_{129}$ | 11.311 | 1.152 | 12 | $10_{130}$ | 12.361 | 0 | 12 |
| $10_{131}$ | 11.05 | 1.678 | 13 | $10_{132}$ | 8.684 | 0 | 10 |
| $10_{133}$ | 18.68 | 0 | 10 | $10_{134}$ | 18.151 | 0 | 11 |
| $10_{135}$ | 16.05 | 0 | 10 | $10_{136}$ | 3.158 | 12.46 | 9 |
| $10_{137}$ | 0.526 | 13.68 | 12 | $10_{138}$ | 2.632 | 15.26 | 10 |
| $10_{139}$ | 16.311 | 0 | 10 | $10_{140}$ | 13.151 | 4.31 | 13 |
| $10_{141}$ | 15.26 | 1.152 | 10 | $10_{142}$ | 13.68 | 0 | 11 |
| $10_{143}$ | 16.311 | 0 | 12 | $10_{144}$ | 10.521 | 7.105 | 11 |
| $10_{145}$ | 9.211 | 0 | 10 | $10_{146}$ | 10.26 | 4.31 | 9 |
| $10_{147}$ | 4.474 | 6.415 | 12 | $10_{148}$ | 17.63 | 0 | 12 |
| $10_{149}$ | 16.84 | 1.678 | 12 | $10_{150}$ | 16.571 | 3.257 | 12 |
| $10_{151}$ | 18.941 | 0 | 11 | $10_{152}$ | 21.05 | 0 | 8 |
| $10_{153}$ | 18.941 | 0 | 11 | $10_{154}$ | 23.151 | 0 | 8 |
| $10_{155}$ | 9.211 | 6.415 | 9 | $10_{156}$ | 10.521 | 1.152 | 10 |
| $10_{157}$ | 11.84 | 0 | 7 | $10_{158}$ | 6.053 | 9.573 | 7 |
| $10_{159}$ | 13.941 | 0 | 10 | $10_{160}$ | 8.421 | 2.731 | 12 |
| $10_{161}$ | 10 | 0 | 10 | $10_{162}$ | 7.105 | 5.789 | 11 |
| $10_{163}$ | 6.579 | 3.257 | 9 | $10_{164}$ | 7.105 | 1.678 | 9 |
| $10_{165}$ | 11.311 | 5.789 | 12 | | | | |

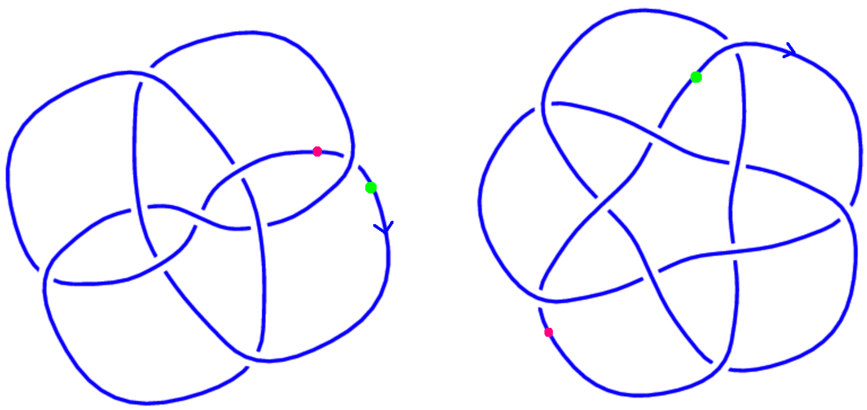

Fig. 6. Both $9_{40}$ and $10_{123}$ knots contain figure-eight knots

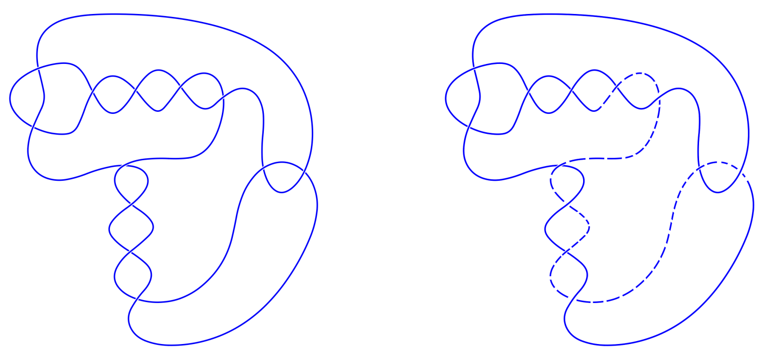

Fig. 7. $12a471$ = [ 2 4 4 2 ] and a $4_1$ complement in $12a471$

The 12 crossing knots without trefoils are shown in Table 6.

The 13 crossing knots without trefoils are shown in Table 7.

The 14 crossing alternating knots without trefoils are shown in Table 8.

We note that the eleven crossing knots that do not contain trefoils but they do contain 7.7's which do contain trefoils. Thus, the subknot relationship is not transitive in the setting of subknots of minimal crossing prime knot presentations.

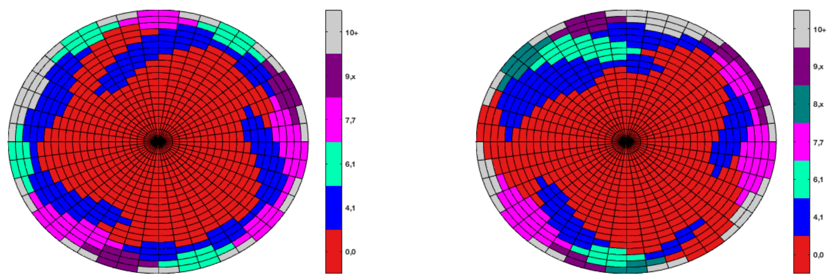


Fig. 8. Both $11a_{135}$ and $11n_{123}$ knots contain figure-eight knots

Table 6: Minimal 12 Crossing Prime Knots without Trefoil Subknots

| | | | |
|---|---|---|---|
| $12a_{125}$ | $[(22;2)(22;2)]$ | $12a_{128}$ | $[24;22;2]$ |
| $12a_{181}$ | $[22;22;2+2]$ | $12a_{183}$ | $[42;22;2]$ |
| $12a_{204}$ | $[23111112]$ | $12a_{265}$ | $[8*21110]$ |
| $12a_{286}$ | $[8*21.20.2]$ | $12a_{448}$ | $[4;22;22]$ |
| $12a_{460}$ | $[2.21.210.2]$ | $12a_{518}$ | $[41111112]$ |
| $12a_{975}$ | $[8*20:20:20:20]$ | $12a_{1022}$ | $[9*20:.20:.20]$ |
| $12a_{1124}$ | $[2.2.2.2.20.20]$ | $12a_{1202}$ | $[2.2.20.2.2.20]$ |
| $12a_{1213}$ | $[101*30]$ | $12n_{11}$ | $[231;-211;2]$ |
| $12n_{18}$ | $[(22;2)(211;2-)]$ | $12n_{298}$ | $[8*-2110:.20]$ |
| $12n_{525}$ | $[21:21:-30]$ | $12n_{556}$ | $[3;21;-21;21]$ |
| $12n_{844}$ | $[8*2.20.2:.-20]$ | | |

Table 7: Minimal 13 Crossing Prime Knots without Trefoil Subknots

| | | | |
|---|---|---|---|
| $13a_{1}$ | $13a_{21}$ | $13a_{563}$ | $13a_{577}$ |
| $13a_{597}$ | $13a_{655}$ | $13a_{757}$ | $13a_{1250}$ |
| $13a_{1415}$ | $13a_{1443}$ | $13a_{1717}$ | $13a_{2428}$ |
| $13n_{1}$ | $13n_{3}$ | $13n_{38}$ | $13n_{63}$ |
| $13n_{87}$ | $13n_{182}$ | $13n_{1354}$ | $13n_{1523}$ |
| $13n_{2014}$ | $13n_{2325}$ | $13n_{2614}$ | $13n_{2767}$ |
| $13n_{2827}$ | $13n_{3155}$ | $13n_{3229}$ | $13n_{3230}$ |
| $13n_{3286}$ | | | |

## 5. Conclusions and Open Questions

These are just the first steps toward a complete analysis of subknots present in small minimal knot diagrams of prime knots. They provide some optimisitic and limited evidence in support of the conjectures. Due to the limitations of our com-

Table 8: Minimal 14 Crossing Prime Knots without Trefoil Subknots

| | | | |
|---|---|---|---|
| $14a_{5}$ | $14a_{69}$ | $14a_{743}$ | $14a_{1424}$ |
| $14a_{1427}$ | $14a_{1450}$ | $14a_{1452}$ | $14a_{1465}$ |
| $14a_{1491}$ | $14a_{1499}$ | $14a_{2075}$ | $14a_{2085}$ |
| $14a_{2116}$ | $14a_{2128}$ | $14a_{2141}$ | $14a_{2160}$ |
| $14a_{2460}$ | $14a_{2552}$ | $14a_{2555}$ | $14a_{2556}$ |
| $14a_{2558}$ | $14a_{2651}$ | $14a_{2666}$ | $14a_{2705}$ |
| $14a_{2757}$ | $14a_{2904}$ | $14a_{2984}$ | $14a_{3195}$ |
| $14a_{3376}$ | $14a_{3387}$ | $14a_{3530}$ | $14a_{3534}$ |
| $14a_{3543}$ | $14a_{4368}$ | $14a_{4437}$ | $14a_{4917}$ |
| $14a_{5072}$ | $14a_{5157}$ | $14a_{5581}$ | $14a_{5697}$ |
| $14a_{5771}$ | $14a_{5773}$ | $14a_{5845}$ | $14a_{5846}$ |
| $14a_{5958}$ | $14a_{5959}$ | $14a_{5973}$ | $14a_{5983}$ |
| $14a_{6002}$ | $14a_{6048}$ | $14a_{6127}$ | $14a_{6166}$ |
| $14a_{6211}$ | $14a_{6404}$ | $14a_{6414}$ | $14a_{6555}$ |
| $14a_{6561}$ | $14a_{6650}$ | $14a_{7185}$ | $14a_{7192}$ |
| $14a_{7271}$ | $14a_{7282}$ | $14a_{7382}$ | $14a_{7447}$ |
| $14a_{7898}$ | $14a_{8233}$ | $14a_{9261}$ | $14a_{9329}$ |
| $14a_{9414}$ | $14a_{9489}$ | $14a_{9585}$ | $14a_{9594}$ |
| $14a_{9599}$ | $14a_{9634}$ | $14a_{9641}$ | $14a_{9866}$ |
| $14a_{10167}$ | $14a_{10415}$ | $14a_{10552}$ | $14a_{11685}$ |
| $14a_{11689}$ | $14a_{11690}$ | $14a_{11859}$ | $14a_{12162}$ |
| $14a_{12741}$ | $14a_{14500}$ | $14a_{14510}$ | $14a_{14525}$ |
| $14a_{14888}$ | $14a_{14926}$ | $14a_{14934}$ | $14a_{14943}$ |
| $14a_{15415}$ | $14a_{16442}$ | $14a_{16480}$ | $14a_{17322}$ |
| $14a_{17385}$ | $14a_{17388}$ | $14a_{17406}$ | $14a_{17730}$ |
| $14a_{18051}$ | $14a_{18053}$ | $14a_{18241}$ | $14a_{18309}$ |
| $14a_{18617}$ | $14a_{18626}$ | $14a_{18723}$ | $14a_{19429}$ |
| $14a_{19478}$ | | | |

putational capacity, we have not completed the analysis of the 16 crossing prime knot presentations. We wonder, if one were to undertake a complete analysis of the already classified prime alternating and non-alternating knot presentations, what would one find? Would there be additional evidence to confirm these conjectures or might one find it necessary to add a new basic knot type to the trefoil and figure-eight types? Nevertheless, based on our analysis to date, one expects to find either trefoils or figure-eight subknots as, in some sense, these are still quite simple presentations.

If the evidence supports the conjectures, then one must face the question of how one might go about constructing a proof. This seems to be a rather complex question if only because we have not been able to envisage a systematic abstract algorithm

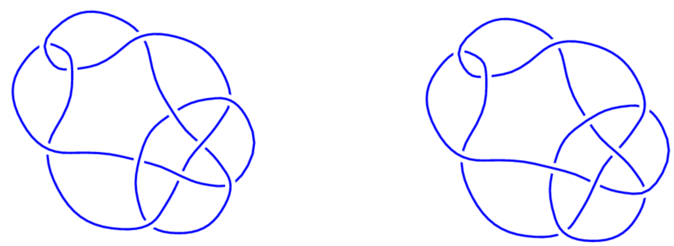

Fig. 9. K11a135 and K11n123

to test the conjecture. We have been able to prove the existence of an elementary infinite family of algebraic knots that do not contrain trefoils but do contain figure-eight knots and have detected other such knots that are not incompassed by this theorem.

The perspective of the 2009 Trieste conversation was the search for a small collection of irreducible knot types from which every prime knot is created, a theory of "elementary knot types." An unexplored dimension related to the presence of these unknots concerns their topological consequences. For example, one reason that a result of this type would be interesting is that it could lead to an elementary strategy by which one could prove that certain knot invariants, e.g. the Jones polynomial, could detect non-trivial knotting. Others would be the implications for the genus of the knot or for non-trivial representations of the knot group.

There are several additional questions that we wish to suggest as some may find then attractive. First, we have looked only at the 'standard' minimal crossing presentations of prime knots. There are, however other minimal crossing presentations of these knots. Thus one lead to ask

**Question: How does the subknot population depend on the minimal crossing presentation?**

Jablan and Millett proposed a very specific definition of the knot type of a segment in a knot presentation [6]. Thus, one is lead to ask

**Question: How does the subknot population depend on the closure identification method?**

For example, Jablan and Millett used the 'above' closure definition as this is a natural perspective of one looking at knot presentations but one might imagine using a 'below' closure. Alex Rich implemented this approach and collected the data through eight crossings for the sake of comparison with our data. He found slight differences for the following knots: $7_6$,$8_8$,$8_{10}$,$8_{14}$,$8_{19}$, and $8_{20}$. The most striking difference is found for $8_{20}$ where the 'up' closure does not find any $4_1$'s but they are present in the 'down' closure.

As the rational knots, also known as the 2-bridge knots, have simple Conway

| Knot type | Up number | Down number |
|---|---|---|
| $0_1$ | 167 | 155 |
| $3_1$ | 38 | 48 |
| $3_1\#3_1$ | 7 | 7 |
| $4_1$ | 0 | 4 |
| $7_6$ | 2 | 2 |
| $8_{20}$ | 10 | 10 |

presentations, form an attractive first class on which to test conjectures, the following questions is attractive:

**Can one identify the entire family of minimal crossing prime 2-bridge presentations that do not contain trefoils and prove they contain figure-eight subknots?**

From the perspective of physical knots, one hopes that the analysis of presentations provides insight into the spatial properties of the knots and is lead to ask

**Question: How do the populations of minimal crossing prime knot presentation subknots compare with the collections of subknots of spatial knots such as the 'KnotPlot" or the 'Ideal' knots studied in [9]?**

More generally, one is lead to ask

**What does the population of subknots of a knot imply about the complexity of the knot [4] or its spatial character?**

Especially,

**What does the subknot structure of a minimal crossing presentation of a knot imply about knot invariants associated to the knot?**

## 6. Acknowledgements

We would like to express our gratitude to Joseph Migler for his early contribution to this project, to Slavik Jablan for his conversations and collaboration in the refinement of the conjectures, to Morwen Thistlethwaite for his assistance in with the knot data and, to Eric Rawdon and Andrzej Stasiak for their collaboration with KM in the creation and application of knotting fingerprints, an analogue of which we have employed in this research. The knot images in Figures 1, 2, 3, 5, 6, and 9 derive from those found in KnotInfo [1].

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