To appear in the *Journal of Mathematics and the Arts*
Vol. 00, No. 00, Month 20XX, 1–31

# *A Mathematical Analysis of Knotting and Linking in Leonardo da Vinci's Cartelle of the Accademia Vinciana*

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(*submitted November 2014*)

Images of knotting and linking are found in many of the drawings and paintings of Leonardo da Vinci, but nowhere as powerfully as in the six engravings known as the *cartelle of the Accademia Vinciana*. We give a mathematical analysis of the complex characteristics of the knotting and linking found therein, the symmetry these structures embody, the application of topological measures to quantify some aspects of these configurations, a comparison of the complexity of each of the engravings, a discussion of the anomalies found in them, and a comparison with the forms of knotting and linking found in the engravings with those found in a number of Leonardo's paintings.

**Keywords:** Leonardo da Vinci; engravings; knotting; linking; geometry; symmetry; dihedral group; alternating link; Accademia Vinciana

*AMS Subject Classification*: 00A66; 20F99; 57M25; 57M60

## 1. Introduction

In addition to his roughly fifteen celebrated paintings and his many journals and notes of widely ranging explorations, Leonardo da Vinci is credited with the creation of six intricate designs representing entangled loops, for example Figure 1, in the 1490's. Originally constructed as copperplate engravings, these designs were attributed to Leonardo and 'known as the cartelle of the Accademia Vinciana' [1]. The Academia Leonardi Vinci is thought to have been a 'casual coming together of cultivated individuals for the purpose of dialog and exchange' [2]. The group was loosely associated with the court of the Duke of Sforza in Milan, and contributed to the era historically proclaimed as Milan's 'Golden Age' [2]. Leonardo da Vinci spent seventeen years in Milan (1482-1499) [3], and it was during that time that he is to have created the six designs of the cartelle. Although each is stunningly symmetric and artistically impressive, the cartelle engravings have been (and continue to be) a matter of some mystery. The general focus among art historians on masterpieces associated with Florence and Rome has served to obscure the cartelle from the public attention [2].

Since their creation, the entangled designs of the cartelle have been surrounded by the misconception that each one is formed with a single line. This error can be attributed in part to Giorgio Vasari, an Italian painter, architect, and writer, famous for his publication of *Lives of the Most Eminent Painters, Sculptors, and Architects* [4] in 1550. He wrote, 'Leonardo spent much time in making a regular design of a series of knots so that the cord may be traced from one end to the other, the whole filling a round space'. Upon studying the cartelle designs, one can easily discern that they are not created with a single curve, but instead, each one is formed by linking several curves together. The misconception, ignited by Vasari's comment, apparently

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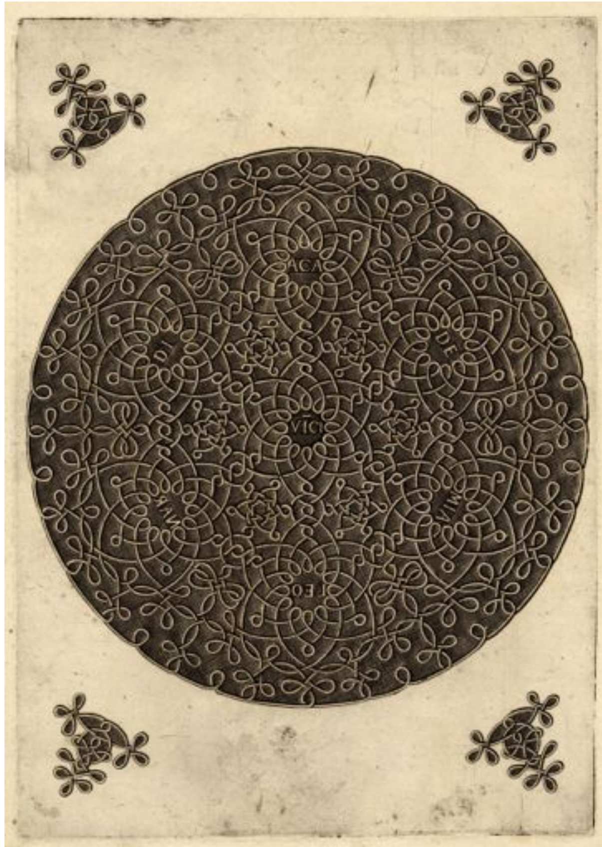

Figure 1. **The sixth of Leonardo‘s six engravings (With permission of the British Museum)**

diffused through scholarship and connected the cartelle engravings with the medieval idea of a labyrinth, or a maze. In reality, the entangled designs are comprised of multiple components that intertwine to create the overall effect of a maze, but are not true labyrinthine representations. These engravings have been the subject of speculation from many philosophically different perspectives. They have been copied by the German artist, Albrecht Dürer in a widely visible series of wood cuts produced in about 1506-7, during Dürer's second visit to Italy. Initially trained as a goldsmith by his father, his unusual talents lead to an apprenticeship with an artist and the study of woodcuts. His first visit to Italy occurred when he was only 23 and already quite famous for his work. The woodcuts of the cartelle provide a quite faithful copy of those of Leonardo, including some of their anomalies, though there are quite subtle changes in local scale that may give insight into their creation. We use several of these in cases where we have been unable to find complete images of Leonardo's engravings, see Figures 20, 31 and 52.

### 1.1 *How we came to study the cartelle*

In 2010, Caroline Cocciardi contacted Ken Millett seeking a mathematical analysis of the embroidery on the Mona Lisa's bodice, Figure 2, whose structure had become more clearly visible with the application of Pascal Cotte's photography able to achieve a resolution of 240 million pixels. Caroline had recently produced a documentary, Mona Lisa Revealed, and was interested in pursuing the implications of the structure found in this embroidery. Though not containing any mathematically knotted features, one observes a significant regularity as well as a violation of the apparently intended regular structure. Caroline shared her passion for the Mona Lisa, in particular, as well as her interest in understanding the presence of 'knotted' features in Leonardo's paintings. In addition, she shared images of the six engravings whose startling degree of complexity and symmetry immediately provokes interest. This made the subject of

Figure 2. **The Mona Lisa showing the bodice detail. Along the upper border one observes a chain of linked circles and, below this, a twisted cord.**

Leonardo's knots a perfect addition to the Spring 201 undergraduate honors seminar on 'Knots in Nature' that Ken was leading at the University of California, Santa Barbara. With this experience, the following fall six honors students, Corrine Cooper, Jessica Hoy, David Hyde, Bryan Lee, and Erica Mason, enrolled in a ten-week honors course focused on learning the underlying mathematical knot theory and applying this knowledge to initiate a topological analysis of the engravings. In Spring 2012, Corrine took an art history class for which she wrote a paper on the cartelle that has enriched our historical understanding of them. In Fall 2012, Jessica and Ken began work on assembling the results of the honors course and extending and completing the mathematical analysis, as well as producing a report on what has been discovered over the course of these two years. This article is a summary of the results of this mathematical analysis.

### 1.2 *What has been learned*

While there are some anomalies, the apparent overall structural regularity is supported by our careful topological assessment of the local structure of each of the individual designs. From a geometric perspective, we see a sufficiently significant level of local variation, and thus we are confident that these engravings were created without the use of a precise template to guide their construction. We observe that one can quantify the visually apparent aggregate symmetry using symmetry groups visible in the structure of the components that constitute the aggregate structure in each of the links. As the basic structural symmetry originates from equilateral triangles and squares, one is lead to wonder why a pentagonal base was not employed, especially because it is often proposed that the engravings were intended to puzzle the viewer. There is a much wider variation in the number of components than one might assume without a careful analysis of each of the engravings.

Figure 3. **The tablecloth knot found in the Last Supper.**

### 1.3 *Organization of our report*

In section 2, we will briefly describe the underlying mathematical concepts employed in our analysis. In section 3, we will provide a description of the overall symmetry and topological elements, followed by our analysis of each individual design in section 4. In section 5, we will describe the anomalies, that is the exceptions to the apparent structural regularity characteristic of the cartelle and finally, in section 6, we will give a concluding overview of what we have discovered.

## 2. The mathematics employed in our analysis of the cartelle

There appears to be no conclusive evidence concerning the methods utilized by Leonardo in the creation of the cartelle links, but one can speculate about methods that may have guided or inspired him in their realization. In her lecture ‘DaVinci’s Knots, Caroline Cocciardi describes her research of Leonardo da Vinci and his use of ‘knots”. She states that in every one of his thirteen masterpieces, Leonardo ‘always leaves an interlocking knot’ [5]. From a traditional mathematical perspective, this assertion requires elaboration as classical knot theory concerns the spatial structure of closed curves in three dimensional space [6]. For many important purposes, this constraint is inadequate and one finds it necessary to provide a mathematical theory that is consistent with one’s common sense notion of knotting encountered in open spatial curves. This has been accomplished through a careful analysis of a closure that successfully captures knotting important in the spatial structure of macromolecules [7]. Although ‘The Mona Lisa bodice, Figure 2, contains only linked chains and an unknotted (at every scale) open twisted cord,‘The Last Supper, Figure 3 (among other da Vinci works) does contain an open knot, here found tied in the right hand corner of the tablecloth.

None of the many knotted conformations found in Leonardo’s paintings come close to the complexity of the structure found in the cartelle designs. Many elements of the cartelle links exhibit similarities to both Celtic Knots and Mirror Curves, suggesting a possible similarity with the Celtic and Tamil construction methods. Art historians agree that the entangled designs were most likely created by Leonardo da Vinci as a puzzle, but were used later in various textile applications. Albrecht Dúrer, a German artist of the Renaissance period most known for his printmaking and woodcut engravings [8], likely acquired a set of Leonardo’s copperplate engravings during his stay in Venice in 1506-07 [2]. He then published a series of six woodcut engravings entitled *Sechs Knoten*, which appear to have faithfully duplicated Leonardo’s designs. Each of Dúrer’s engravings is an exact replica of the corresponding cartelle link, with minor embellishments of the ancillary corner components, and an omission of the *Academia Leonardi Vinci* inscription in the center. We have analyzed both the original and Dúrer copies of the

Figure 4. **The dihedral group, $D_8$, of the octagonal stop sign.**

six entangled designs of the *cartelle*, sometimes referred to as *medallions*, from a mathematical perspective as well as reviewing the historic and artistic commentary in the hope of developing new illuminating information that might deepen our understanding and guide further study. For example, historically the designs were often discussed under the assumption that each one consisted of a single uninterrupted cord. This misconception led some to the belief that each design was a representation of a labyrinth, an ancient concept that became popular during the Renaissance. The labyrinthine idea was then connected to notions of femininity and possessed deep philosophical and religious significance. These views were undermined as the consequence of later observation that each of these did not, in fact, consist of a single component (i.e. a individual closed strand) but rather, each were composed of differing numbers of components. The number of individual components in each of the designs is just the first possible mathematical feature that can inform one's understanding of the nature of the medallions.

## 2.1 *The mathematical foundations of our analysis*

We employ the mathematics of algebra, geometry, and topology to analyze each of the six medallions of the cartelle. Though interconnected, each of these facets of mathematics captures distinct important structural features that will play essential roles in our analysis. To set the stage for the discussion of our analysis, we will review a few essential elements of each of these areas.

### *2.1.1 Algebraic considerations*

The powerful initial impression of the engravings is the thematic presence of symmetry. While each design displays a distinct pattern of knotting and linking with a varying number of components, each of these components, in themselves and collectively, display varying degrees of symmetric spatial structures contributing to the stunning degree of complexity and regularity displayed in the aggregate structure. The three dimensionality portrayed in the engravings is fundamental to understanding and quantifying the nature of the symmetric spatial structure. In this context, symmetry means the existence of a three dimensional rigid motion taking an object exactly to itself thereby defining its group of symmetries [9]. In this case, the object is the spatial conformation of curves portrayed in a medallion. In order to make concrete the mathematical concepts we require, we first consider the octagonal 'Stop Sign' shown in Figure 4. One facet of the Stop Sign symmetry is that its placement will be unchanged by a $\frac{2\pi}{8}$ rotation in a counter-clockwise direction in the plane of the image and about its center. This rotation can be repeated, so that along with the identity (no rotation at all), one defines a set of eight transformations known as the *rotation group* whose motions are shown in the first line of Figure 4. There is, however, another symmetry given by reflection across an edge, the effect of which is to change the 'orientation' of the edges of the octagon as suggested by the changed image of the word 'Stop.' The addition of this reflection and its composition with the rotations double the number of transformations in this group of symmetries to sixteen and defines the classical *dihedral group*, $D_8$.

In the present context of Leonardo's medallions, we will see that the effect of a mirror reflection is a change in the topological identity of the spatial structure. Nevertheless, the relevant group

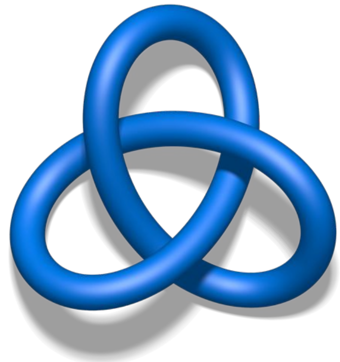

Figure 5. **The negative trefoil knot, -T, $-3_1$ in the Alexander-Briggs classification.**

of symmetries is still the dihedral group formed by replacing the planar reflection with a three space rotation. This is illustrated by considering the *trefoil knot* shown in Figure 5. Here one observes invariance under rotations by $\frac{2\pi}{3}$ giving a three element rotation group. Notice that one may rotate the trefoil about the 'vertical" axis in the plane of the image passing through the two upper strands and between the lower crossing strands by $\pi$ to take the trefoil to itself but reversing the direction along the spatial curve. As a rigid spatial motion, this has not changed the type of knot (as a mirror reflection would have done), thereby giving a new symmetry which, when added to the rotations, creates a three dimensional representation of the dihedral group, $D_3$.

We will see that dihedral groups of exactly this type quantify the symmetry found in Leonardo's medallions and their constituent components giving concrete substance to the degree of three dimensional symmetry they manifest. For example, these significant manifestations of symmetry enable one to factor the aggregate structural symmetry into subfamilies that exhibit this symmetry and contribute to the specific symmetry of the whole. We pose two questions: (1) What is the source of the desire to attain these certain symmetries in the atomic conformations? (2) What is the nature of their contributions to the whole? We observe that the symmetry groups of the designs are all dihedral groups, reflecting the apparent circular symmetry of each of the links. However, as noted, these arise in a manner significantly different from the usual dihedral groups in which two dimensional or planar reflections through an axis of symmetry classically play the role of one of the generators. Leonardo's dihedral symmetry is one that is fundamentally three dimensional in nature in that the second generator is given by a three dimensional rotation about an axis of symmetry. With the exception of rare errors in the design, one striking feature of the structure is the alternation of under and over crossing when traversing the path of each of the constituent cords in the aggregate structure. The three dimensional symmetry is required to preserve the alternating nature and chiral structure of the conformations, as the normal planar reflection across an axis would invert the over and under crossings, and thereby change the three dimensional structure to an inequivalent one as we would see when taking topology into consideration.

#### *2.1.2 Geometric considerations*

In this context, the fundamental notions of *geometry* that will concern us pertain to those of curves in space, i.e. spatial distance and length along a curve, measures of curvature or bending, and twisting or torsion [10]. While assessing the group of the symmetries apparent in each of Leonardo's engravings, one wonders how such a complex design could have been produced. For example, did Leonardo create a template for a portion of the design having periodic boundary structure that allowed him to use the template to realize the symmetric structure? One feature of the use of such a template would be the exact reproduction of the local geometry in each of the locations corresponding to a specific feature. This can be tested by employing local geometric measurements such as the length of the curve between crossings or the curvature in the bending

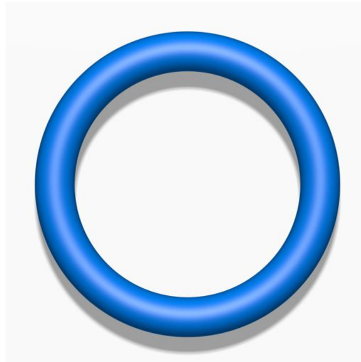

Figure 6. **The trivial knot is any conformation which can be lambently deformed in space to the standard circle pictured here.**

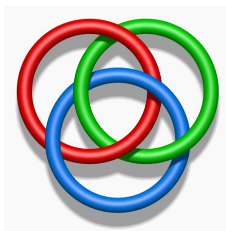

Figure 7. **The Borromean Rings. This Wikipedia image is considered to be in the public domain.**

Figure 8. **The Borromeo family crest [24]**

inherent to an individual loop, as seen in the trefoil, Figure 5. These fundamental quantities must be preserved under rigid symmetry as well as being reflective of the use of a template in creation of the structure.

#### *2.1.3 Topological considerations*

Because each emblem of the cartelle consists of multiple strands, they are mathematically classified as links, i.e. consisting of a collection of non-intersecting spatial closed curves. The case of a one component is called a knot: the trivial knot (Figure 6) and the trefoil knot (Figure 5), are the simplest examples. Colin Adams [6] describes a link as ‘a set of knotted loops all tangled up together’. The Borromean Rings, shown in Figure 7, provide a familiar example of a three component link. The link consists of three interlocked rings, and is of special mathematical interest because while no two rings are linked to one another, the union of the three is inseparable. Note that this image suggests that it can be realized by three standard circles. This is, however, not true. This image is a popular illusion. The rings are named for the Borromeo family of fifteenth century Milan. The symbol was a ‘gift from Francesco Sforza as recognition of the support that the Borromeo family had given in defense of Milan” [24]. The rings can be seen on the Borromeo family crest (Figure 8) in the bottom right corner of the left half. While, in this case, each of the components is a trivial knot, this is not always the case. We catalog Leonardo’s six medallions as *Link 1* - *Link 6*, see Figures 20, 31, 41, 52, 67 & 80. Due to the complexity of these conformations, we will require several distinct measures of topological structure in order to describe their individual character in a manner that will support a comparative analysis.

Classical knot theory defines the *linking number* between two oriented components to provide a tool with which to quantify the linking of two oriented curves. To compute this linking number, one must first assign an orientation to each of the curves in the link, i.e. a direction on each of the curves. A generic projection of the spatial conformation contains a finite number of places where one component crosses the other. Each crossing of the two different components is then designated as a positive or negative crossing; positive if the curve on the top comes from the left, see Figure 9, and negative if the curve on the top comes from the right, see Figure 10. The

Figure 9. $+1$ **crossing**

Figure 10. $-1$ **crossing**

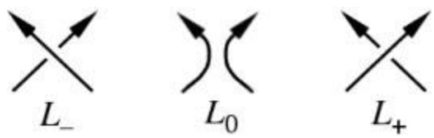


Figure 11. **At a single crossing in a presentation of an oriented link, L, one makes two changes to create a family of three links. These enable the fundamental relation defining the HOMFLY-PT polynomial.**

sum of all the positive and negative crossings, divided by two, gives the integer linking number of the two oriented curves. If one adds only the self-crossings of a single component, the result is called the *writhe* of the given presentation. While the linking number of two oriented curves is a topological invariant, unchanged under ambient isotopy equivalence of links, the writhe is not a topological invariant. We calculate the linking numbers of some of the links or sublinks in the cartelle as an aid in uncovering fundamental structural information.

Each design of the cartelle consists of a principal link as well as four ancillary knots or links in the corners. We will want to identify and categorize these smaller components as well as the motifs found in the more complex design. This will be accomplished, where possible, using a contemporary knot invariant, the *HOMFLY-PT* polynomial P(L) [13], of an oriented link, L, to identify the conformation using the Alexander-Briggs [17] or Hoste-Thistlethwaite classifications [14]. The HOMFLY-PT polynomial is the unique two-variable, $\ell$ and $m$, extension of the 1923 Alexander [11] and the 1985 Jones [12] polynomials enjoying the following properties: (1) P(unknot) = 1 and (2) if three link presentations, $L_+$, $L_-$, and $L_0$ differ at a single crossing as shown in Figure 11, then $\ell$ P$(L_+)$ + $\ell^{-1}$ P$(L_-)$ + $m$ P$(L_0) = 0$. By appropriately changing crossings one may use these properties to determine the HOMFLY-PT polynomial of any oriented knot or link and, under most circumstances, identify the given knot [15].

For example, the HOMFLY-PT polynomial of the negative trefoil knot, -T, shown in Figure 5, is

$$\text{P(-T)} = -2\ell^2 - \ell^4 + \ell^2 m^2.$$

The positive trefoil, +T, is defined by reversing each of the crossings in the presentation. One can see from the HOMFLY-PT polynomial definition, that the calculation of $P(+T)$ will follow the same steps but with $\ell$ replaces by $\frac{1}{\ell}$ to give

$$\text{P(+T)} = -\ell^{-4} - 2\ell^{-2} + \ell^{-2} m^2.$$

From the fact that these two equations are different, one sees that the '-' and '+' trefoils are topologically distinct, i.e. it is impossible to ambiently evolve one trefoil to the other. This is typical of how one can apply a polynomial knot invariant to the analysis of a conformation and in comparisons between conformations.

The Jones polynomial can be determined from the HOMFLY-PT polynomial by setting $\ell = it^{-1}$ and $m = i(t^{\frac{-1}{2}} - t^{\frac{1}{2}})$ from which one finds that the Jones polynomial of -T is $-t^{-4} + t^{-3} + t^{-1}$ and that of +T is $t^1 + t^3 - t^4$.

Due to the complexity of the medallions, we often wish to compare portions of the designs and, thereby wish to apply a type of knot theory that works for open ended cords that are frozen in position (see Figure 12). From a classical topological perspective, such conformations are unknotted in that they can be evolved into a collection of straight segments. Nevertheless, the theory of such knots has proved to be critical in the analysis of, for example, protein structures [16]. Our analysis will make use of those open knots as they arise as local portions of the constituent components of the global structure. We will also use the analogously frozen version

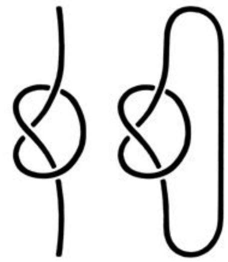

Figure 12. **The open overhand knot becomes the minus trefoil, -T, when its ends are closed as shown [25]**

of *tangle* theory involving collections of open segments naturally lying in a selected region. Elements of tangle theory naturally arise in the study of knots and knot polynomials [6, 15].

## 3. The mathematical analysis of the cartelle

In our analysis of the cartelle we will consider the central designs as well as the corner links. The latter contain some of the same local elements, but are overall much simpler than the central entangled designs. Often, these ancillary components consist of only one strand, and are thus one component links, i.e. knots. The complexity of each central emblem results from the number of its components, from the symmetry group of each of the individual components, collections of components, and the entire design, and from the structure of the entanglement of the individual components and the associated consequences for the symmetry of the aggregate structures. Among the six designs, we identify two dominant structural classes, *radial* and *circular*, although each exhibits both to some degree. The dominant radial structure (seen in Links 1, 2, and 3) is characterized by a presence of concentric rings of different shapes. When viewing these emblems, one's attention is drawn to the different radial layers of local knots or links (Figures 20, 31, 41). These conformations have symmetry group $D_{16}$. We note that the conformations that exhibit the radial structure appear, visually, to be simpler than those that belong to the circular structural category. In the latter, the eye is drawn to a central circular design surrounded by a number (reflecting different orders of symmetry) of identical designs. Alternating with the outer circular designs, one observes a number of smaller linked components. In the case of Links 4 and 6, the circular structure exhibits $D_6$ symmetry and thus the outer circular designs form a hexagon around the center (Figures 52, 80). This structure appears to be more complex than the radial structure because the central and outer circular designs are constructed in different ways. They are formed by the linking of different components yet result in designs that are locally identical. The fifth conformation has a dominant circular structure despite having a symmetry group $D_8$, a subgroup of the $D_{16}$ symmetry that is characteristic of the radial structures. When viewing Link 5, the eyes are drawn to eight circular designs around the outer portion of the design, as well as a central circle (Figure 67). Contrary to the circular structure defined above, the outer circle is different from the center circle and the two exist as concentric rings much as in the radial structure.

### 3.1 *Common elements*

With the exception of a small number of anomalies, all of the designs exhibit an alternating motif. That is, as one traverses any of the cords in any of the diagrams one alternates between going over and going under at each successive crossing. Though they dominate the classical prime knots with few crossings, the alternating knots and links become exponentially rare with increasing numbers of crossings in the minimal crossing representations within the knot or link

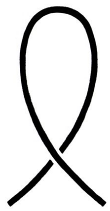

Figure 13. **curl**

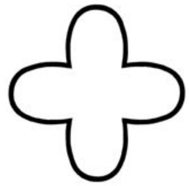

Figure 14. **quatrefoil**

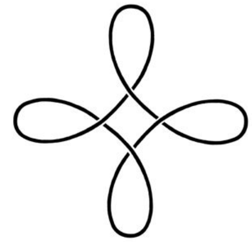

Figure 15. **twisted quatrefoil**

type. In addition, this class enjoys very special properties that have made them an attractive target for topological analysis and the focus of theorems. Why then, one wonders, did Leonardo choose to so limit his designs to those that are alternating when, in nature, one more often sees non-alternating conformations? Specifically, if the objective was to create the most puzzling conformations, one must wonder why non-alternating conformations weren't employed, at least in a few cases.

Several thematic motifs, or subdesigns, and small knots are repeated throughout all six of the cartelle emblems. Most striking is the large numbers of *curls* that one could imagine simply untwisting (see Figure 13) to simplify the structure. These curls are seen in the Mona Lisa bodice (see Figure 2) and other paintings and are ubiquitous in the cartelle. While they surely contribute to the apparent complexity of the design and to the writhe of the conformations, they are topologically irrelevant. In classical knot theory, these elements are called nugatory crossings. They occur frequently in the total conformation due to the apparent symmetry requirements. We note that these curls are common in the symbols of many cultures; they appear to be present widely in geometric figures of the Middle Ages, and there is often a connection to the Christian fish symbol, which has its roots in the early days of the Christian religion.

There is another component that is used abundantly in five of the six emblems. We call it the *twisted quatrefoil*, Figure 15. This figure can be described as a connected sum of four Christian fish symbols, Figure 13, and it is similar to the symbol known as a quatrefoil, Figure 14, prevalent in Gothic architecture and elsewhere in traditional Christian symbolism. This prevalent component may also be described as a simple example of a mirror curve [18]. We note one example of the twisted quatrefoil we have seen in a mosaic floor tile in the Basilica of Aquileia [20] in northern Italy. The twisted quatrefoil can be seen in the upper right corner of Figure 16. The Basilica was built in 1031 on the ruins of the previous structure destroyed in 452 AD. The existing structure today has pieces originating from between the 11th and 17th centuries, so it is difficult to say when the mosaic in the image was created. Some of the mosaics in the basilica even date from the 4th century.

The designs also incorporate familiar knots such as the trefoil, Figure 5, the figure-eight knot, Figure 17, and versions of the endless knot Figure 19 (mathematically known as the $7_4$ knot, Figure 18), among others. One must wonder if designs such as these were also cultural symbols familiar to Leonardo.

## 4. Analysis of individual engravings

In this section we will carefully describe the individual characteristics and properties of each of the six corner and the six principal links of the cartelle and, to the extent possible, provide an overall characterization of each of them. The principal links 1, 2, and 3 exhibit the radial structure as defined above and are members of the $D_{16}$ aggregate symmetry group family. The principal links 4 and 6 exhibit the circular structure, and both have aggregate $D_6$ symmetry group. The principal link 5 demonstrates both the radial and circular structures and has $D_8$ aggregate symmetry group. We will describe the symmetry of the respective components of each

Figure 16. **Mosaics in the Basilica of Aquileia, Italy including a twisted quatrefoil in the upper right corner [21].**

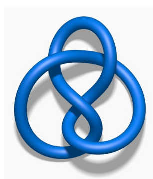

Figure 17. **Figure-eight knot, $4_1$, KnotAtlas [14].**

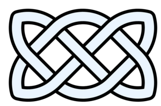

Figure 18. **The $7_4$ knot, KnotAtlas [14].**

Figure 19. **The endless knot.**

conformation and their contribution to the overall symmetry. We will also describe the particular knots and links that we have identified in each emblem in the cartelle.

### 4.1 *Analysis of Link 1*

The first engraving, Link 1, is shown in Figure 20. This is Dürer's copy including his added embellishments and provides an excellent starting point for our analysis as we will see that it exhibits a rich mathematical structure.

#### *4.1.1 The corner configurations*

The *Link 1* corner configuration, Figure 21, is related to the *endless knot*, Figure 19, or topologically, the $7_4$ knot in the Alexander-Briggs enumeration. Many contemporary knot theorists will also know the endless knot as the knot found on the cover of the early editions of Dale Rolfsen's celebrated 1976 book, 'Knots and Links,' [19], Figure 22. The difference between the corner knot in Leonardo's design and its familiar cousin is the absence of one of the two curls or nugatory crossings typically found in presentations of the endless knot, Figure 19. The endless knot is one of the Eight Auspicious Symbols, and thus holds significance in Tibetan Buddhism. Wikipedia [22] states that it represents 'the interweaving of the Spiritual path, the flowing of Time and Movement within That Which is Eternal."

#### *4.1.2 The central link*

Link 1 has six components shown in Figure 23, one outer ring (Figure 24), four middle components (Figure 25), and one inner ring (Figure 26). The middle components connect the outer and inner rings. Their symmetry groups are $D_{16}$, $D_{16}$, and $D_4$ respectively. Each of the four identical middle components exhibits $D_4$ symmetry but, when all four components are placed together, the global $D_{16}$ symmetry is created (Figure 27). The four isomorphic middle components are each a symmetric connected sum of four copies of the knot $8_{18}$, and differ from the adjacent

Figure 20. **Dúrer's version of Link 1**

Figure 21. **The corner design of Link 1**

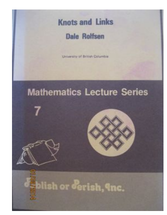


Figure 22. **1976 Cover of Rolfsen's 'Knots and Links'**

copies by a $\frac{\pi}{8}$ rotation. Individually, each middle component is topologically unlinked from the outer ring.

When you look at two adjacent middle components, the outer ring is linked in a manner reminiscent of the Borromean rings, Figure 28. We take a closer look at the local structure of the outer ring and two adjacent middle components (Figures 29 and 30) and find that it forms a three component, eleven crossing alternating link with the following Jones polynomial: $-t^{-5} + 5t^{-4} - 13t^{-3} + 23t^{-2} - 31t^{-1} + 37 - 36t + 33t^2 - 23t^3 + 15t^4 - 6t^5 + t^6$. We note that the alternation of signs of the coefficients is in concert with its minimal alternating crossing presentation. The spread of the degrees is eleven, equal to the number of crossings. Using the Jones polynomial and the assistance of Morwen Thistlewaite, we identified this link as L11a520 [14], the $520^{th}$ link in the Hoste-Thistlethwaite classification of 11 crossing prime alternating links. In this local three-component link, the rings from the adjacent middle components are in fact linked to one another, but the piece from the outer ring is unlinked to either of the rings (Figure 30). If one critical crossing were changed, any two of the three components would be unlinked from one another, individually, and the three pieces would all be topologically linked as a three-component link. With this critical crossing changed, the local structure would thus resemble the traditional representation of the Borromean rings, with a slight embellishment in the linking of the top ring. This local structure in Link 1 could thus be one of the many variations of the traditional symbol.

The outer ring of Link 1 is topologically unknotted, consisting only of an alternating sequence of 16 single and 16 double curls, giving $D_{16}$ symmetry (Figure 23). The inner ring of this link is a $(3, 16)$ torus knot [6], which is linked with the middle components (Figures 26 and 27). To achieve the $D_{16}$ symmetry, this single component must wind around the center in a number of turns that is relatively prime to 16. Three is the smallest such number. The union of this torus

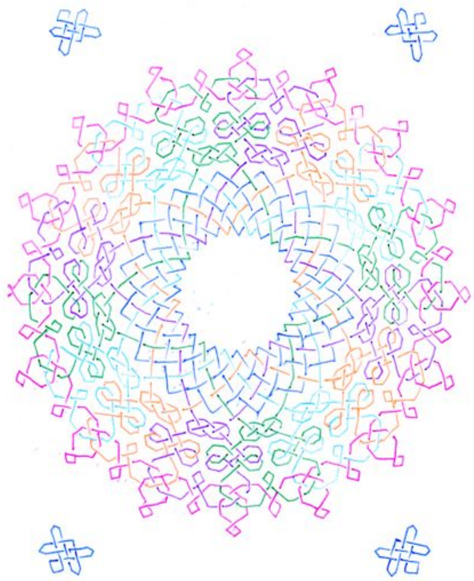

Figure 23. **The six components of Link 1, each identified with a distinct color.**

Figure 24. **The outer ring of Link 1**

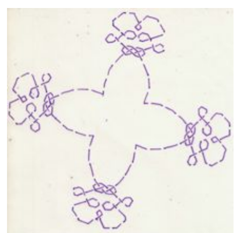

Figure 25. **One of the four middle components**

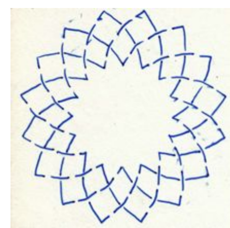

Figure 26. **The center** $(3, 16)$ **torus knot**

Figure 27. **The union of the four middle components of Link 1**

knot with the four interlocking components defines the central structure. Following from the symmetry, we note that the linking of this torus knot with the four middle components is 48.

## 4.2 *Analysis of Link 2*

While Link 1 has an 'angular" appearance, i.e. consisting of straight line segments and sharp angles, Link 2 is rounder with smoothly turning curves in its design and appears to be less

Figure 28. **Location of local linking**

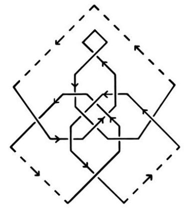

Figure 29. **Borromean like linking in Link 1**

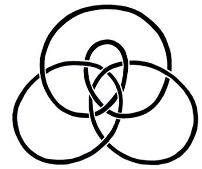

Figure 30. **The 7 crossing link**

Figure 31. **Dúrer's version of Link 2**

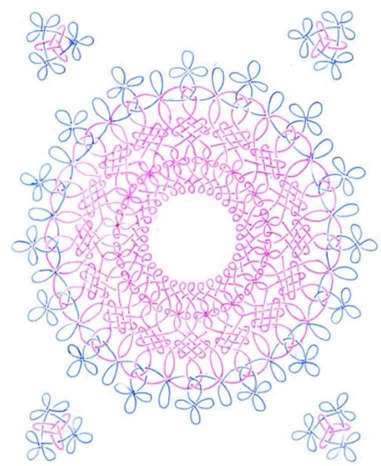

Figure 32. **Link 2 has two components**

complicated, or at least less dense (Figure 31). Indeed Link 2 has only two components, as seen in Figure 32.

### *4.2.1 The corner configurations*

The Link 2 corner configuration (Figure 33) also contains two components, and is the only ancillary figure in the cartelle that is a link instead of a knot. Three of the four (all except the lower right corner) are alternating links. Each is the union of an unknotted component and a negative trefoil knotted component, and is the mirror reflection (i.e. all crossings are reversed) of the link 'L9a32" in the Hoste-Thistlethwaite enumeration [14]. They are topologically equivalent, by removing the visible nugatory crossings, to the mirror reflection of the configuration known as the triquetra entangled with a circle which is a symbol associated with the Christian Trinity (Figure 34). Much art in the Middle Ages was associated with the Christian church. A triquetra consists of three entangled semicircles, and is often described as consisting of three Christian fish symbols, another reference to the notion of the trinity. Note that this is an illusion, much like the Borrmean ring shown in Figure 7, in the sense that it cannot be actually constructed from arcs of circles in the manner suggested by the image. One might wonder if there are other cultural or religious symbols embedded in the knots and links of the cartelle. The triquetra is also known as a trefoil, and is ubiquitous in the study of knots and topology because it the simplest possible knot, aside from the unknot. The lower right corner configuration (Figure 35) is not an alternating link. It is still consists of an unknotted component and a trefoil knotted component, but in this case the trefoil is positive instead of negative as are the other three. This anomalous corner is the link 'L9n15" in the Hoste-Thistlethwaite enumeration [14] and one of very few instances in which the conformations are not alternating.

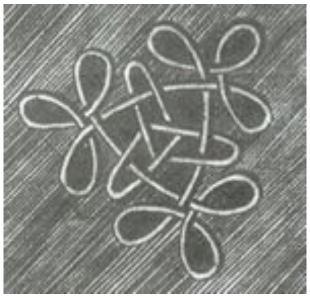

Figure 33. **Corner 2**

Figure 34. **Corner 2 is topologically the same as the triquetra**

Figure 35. **The anomalous lower right corner in Link 2**

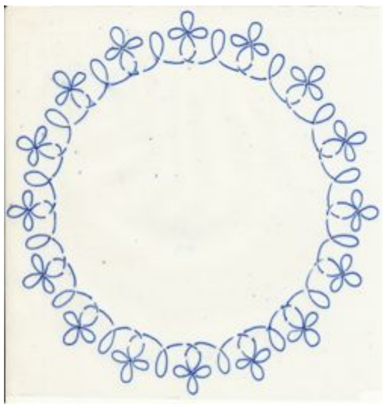

Figure 36. **Outer component of Link 2**

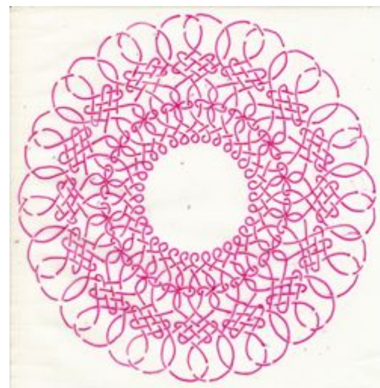

Figure 37. **Inner component of Link 2**

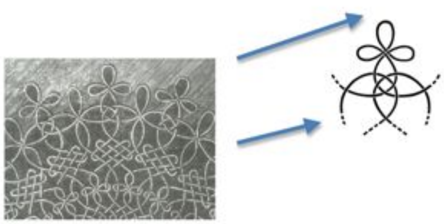

Figure 38. **Location of local linking**

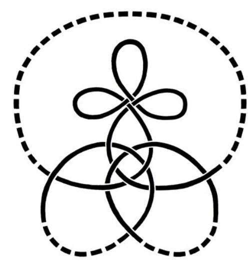

Figure 39. **Borromean similar to the linking in Link 2**

### *4.2.2 The central link*

Link 2 has only two components (Figure 32): one outer ring, and one inner component (Figures 36 and 37, respectively). The component symmetry groups are both $D_{16}$. Although this emblem has the fewest number of components, it is not necessarily the simplest design. As described above, several factors contribute to the complexity of each knot. No single characteristic defines the complexity or simplicity of all the designs, but rather it is a combination of characteristics. The core component of Link 2 contains many complex local knots thereby increasing the overall complexity.

The outer ring of Link 2 is topologically unknotted, and consists of alternating curls and quatrefoils, similar to the cord in the bodice of the Mona Lisa (Figure 2). Indicative of the $D_{16}$ symmetry there are 16 curls and 16 quatrefoils. The inner component makes up the majority of the medallion, and is comprised of concentric rings of different shapes and links. It defies local resolution as it is a prime knot, a direct consequence of Menasco's Theorem [23] on alternating knot and link presentations. There are 32 removable curls that outline a circle in the center. Forming a ring around this circle are 16 copies of a twisted quatrefoil. Just outside the quatrefoils,

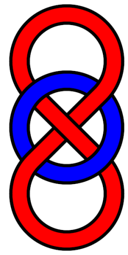

Figure 40. **The Whitehead link, L5a1**

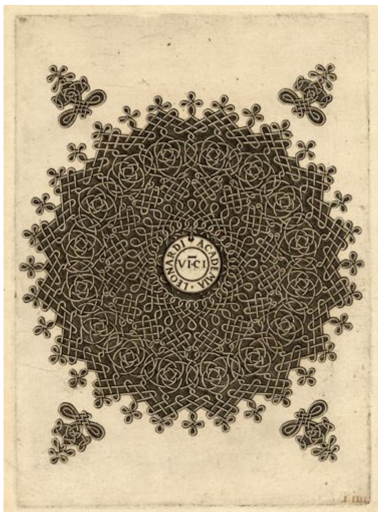

Figure 41. **Link 3 (With permission of the British Museum)**

Figure 42. **Link 3 has six components**

is a ring composed of 16 $7_4$ knots. We see that these are identical to the endless knots in Link 3 (Figure 41), aside from the fact that they are formed from a single component rather than from the amalgamation of two different pieces. The two components in Link 2 are algebraically unlinked (i.e. their algebraic linking number is zero), but they appear to be topologically linked in a manner analogous to, but much more complex than the Whitehead link, Figure 40. Looking at the local structure of the linking between the two components suggests another instance of a Borromean imitation. Considering the outer ring and the two looping pieces of the middle component that link it (Figures 38 and 39), we discover the same three component link as in Link 1, but with opposite sign crossings. While the local link is the same in both instances (Figure 30), the Borromean-like structure of Link 2 differs because the bottom two rings whose linking undermines the full expression of the Borromean rings actually belong to the same global component. We conjecture, from this instance of Borromean mimicry, that while the two components of Link 2 are algebraically unlinked, they are actually topologically linked. As a consequence, it is plausible that the $D_{16}$ symmetry propagates this local structure such that the two components are indeed topologically linked. The global linking is so complex that, at this point, a rigorous proof of the topological linking would seem to require a new or stronger method than that employed to show the Whitehead link is topologically linked.

### 4.3 *Analysis of Link 3*

The symmetry and composition of Link 3 (Figure 41) resembles that of Links 1 and 2 (Figures 20 and 31) in the continuation of the radial nature of their composition featuring prominent concentric rings of motifs. As is the case of Link 1, Link 3 has six components (Figure 42).

Figure 43. **The Link 3 corner is a composite 32 crossing knot**

Figure 44. **The outer ring of Link 3**

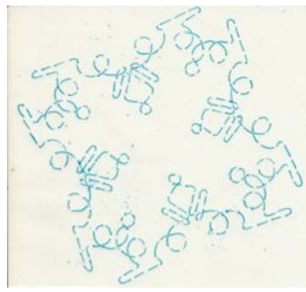

Figure 45. **One of the four middle components of Link 3**

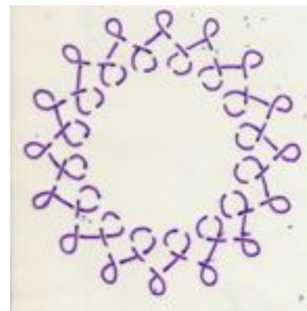

Figure 46. **The central component of Link 3**

### *4.3.1 The corner configurations*

The Link 3 corner (Figure 43) is a 40 crossing alternating composite knot that has 32 irreducible crossings once the visible nugatory crossings are removed. Topologically, it consists of two left-handed trefoils and a core prime alternating 26 crossing knot. The Dowker—Thisthlethwaite [14] code of this prime alternating knot is: 4 10 20 22 2 28 36 38 24 8 6 26 18 52 12 46 48 50 44 32 34 14 16 40 42 30 and its Jones polynomial is: 0 26 : $-1$ 10 $-54$ 206 $-609$ 1481 $-3077$ 5605 $-9097$ 13340 $-17825$ 21840 $-24648$ 25677 $-24716$ 21972 $-18004$ 13550 $-9315$ 5801 $-3233$ 1585 $-666$ 230 $-61$ 11 $-1$. The code for the Jones polynomial indicates that the powers of 't" vary from 0 to 26 with the coefficients of each power listed in increasing order. Thus the constant, $0^{th}$ power term is $-1$. Observe that the terms of the DT code are all positive and that the signs of the coefficients of the Jones polynomial alternate in sign. These are a consequence of that fact that one has an alternating knot or link. The DT codes and Jones polynomials also serve to demonstrate that various knots and links are distinct. Thus, when possible, we include this information in order to better identify the precise knots or links that occur in these configurations.

### *4.3.2 The central link*

Link 3 has six components: one outer ring (Figure 44), four middle components (Figure 45), and one inner ring (Figure 46). Their symmetry groups are $D_{16}$, $D_4$, and $D_{16}$, respectively. The middle components connect the outer and inner rings. Each of the four identical middle components exhibits $D_4$ symmetry. When all four components are placed together, the global $D_{16}$ symmetry is preserved (Figure 47).

In this third link, the outer ring is a connected sum of 16 positive trefoils. Aside from the trefoils, there are looping pieces on the edge that we identify as twisted quatrefoils (Figure 14). Individually, the middle components are topologically unknotted. Each one is linked to the outer ring with a linking number of 16, and to the inner ring with a linking number of 12, resulting in a linking between one middle component and the inner and outer rings of 28. There are two pairs of middle components that differ from each other by a $\frac{\pi}{4}$ rotation (Figure 48); each one

Figure 47. **The union of the four middle components of Link 3**

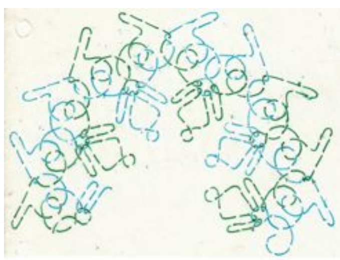

Figure 48. **The middle linking in Link 3**

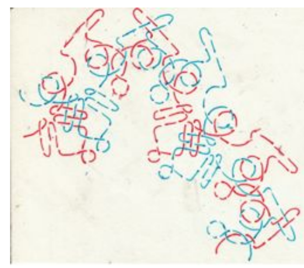

Figure 49. **The middle linking in Link 3**

is linked with linking number 28. There are two pairs of middle components that differ from each other by a $\frac{\pi}{8}$ rotation (Figure 49); each one is linked with linking number 36. The sum of all the linking between the six components gives a total linking number of 312. This number is surprising because it is not evenly divisible by 16, and thus might not seem to be consistent with the 16-fold rotational symmetry. The center ring is topologically unknotted and consists of 32 curls: 16 on the inside, and 16 on the outside of the ring (Figure 46).

As noted previously, Link 3 exhibits the radial structure (i.e. consists of concentric rings of different shapes). The center consists of a ring of intertwined curls from the middle components and inner ring. Outside of this band lie 16 copies of a component that resembles the endless knot (Figure 18). The local endless knot components are woven together by two middle components that differ from each other by a $\frac{\pi}{8}$ rotation (Figure 50). Smoothing the crossings and looking at the piece locally, we discover the endless knot that also appears in Link 2 and in the corners of Link 1 (Figure 21). The ring just outside of the endless knot components consists of 16 copies of a component that is an intertwining of all four middle components. Smoothing the crossings in one of these local conformations results in a design of circles precessing around a center point (Figure 51).

### 4.4 *Analysis of Link 4*

Link 4 (Figure 52) and the other remaining designs, have a large number of components arising from the repetition of individual components (six or eight fold, naturally collected into families depending on the symmetry group). Link 4 comprises 38 components (Figure 53) that together give the aggregate $D_6$ symmetry group.

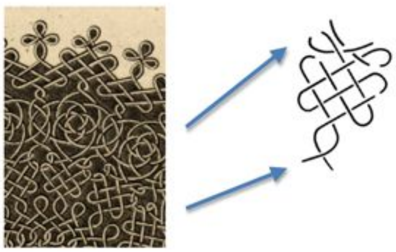

Figure 50. **Woven middle components in Link 3**

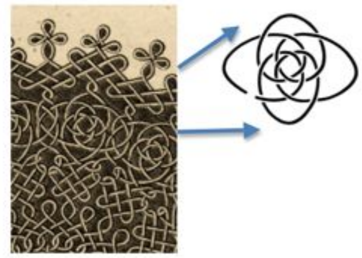

Figure 51. **The precessing circles in Link 3**

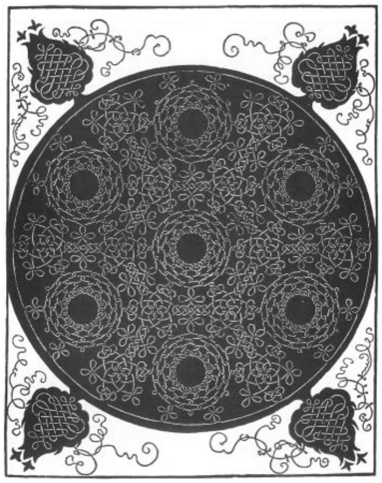

Figure 52. **Dúrer's version of Link 4**

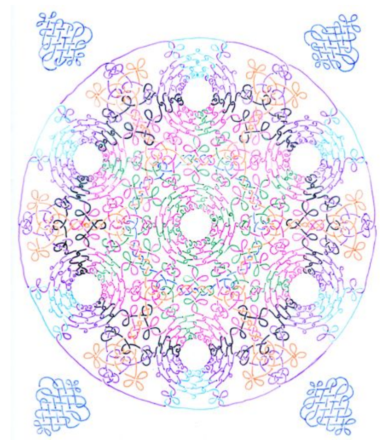

Figure 53. **Link 4 has six components**

Figure 54. **Corner in Link 4**

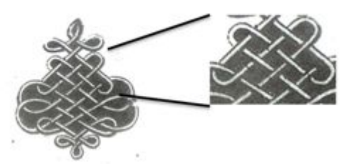

Figure 55. **The corner of Link 4 contains an endless knot**

### *4.4.1 The corner configurations*

The Link 4 corner (Figure 54) has 36 crossings, but after removing the eight visible nugatory crossings, one sees that it is a prime alternating 28 crossing knot. Its DT code is: 4 12 26 56 18 2 8 50 20 10 14 28 36 6 48 40 44 24 46 32 52 30 38 34 54 16 42 22 and its Jones polynomial is $-35 - 7 : 1\ \ -12\ \ 76\ \ -335\ \ 1136\ \ -3116\ \ 7143\ \ -14038\ \ 24146\ \ -36948\ \ 50939\ \ -63872\ \ 73325\ \ -77397\ \ 75303\ \ -67613\ \ 56034\ \ -42846\ \ 30183\ \ -19543\ \ 11584\ \ -6247\ \ 3040\ \ -1316\ \ 499\ \ -160\ \ 42\ \ -8\ \ 1$. In the upper portion of knot, excluding the nugatory crossings, one can see a segment that is the mirror reflection of the endless knot theme found in the corner of Link 1 (Figure 55).

### *4.4.2 The central link*

There are five unique components, rotated and repeated to achieve the total of 38 components. Two different components make up the outer ring (Figures 56 and 57); each component individually possesses $Z_2$ symmetry (given by 3-dimensional rotation about a central axis) and is repeated six times. Each component differs from the previous one by a $\frac{\pi}{3}$ rotation about the center of the medallion, giving rise to the $D_6$ symmetry. The inner ring and another central body component (Figures 58 and 59) are unknotted configurations each also having $D_6$ symmetry.

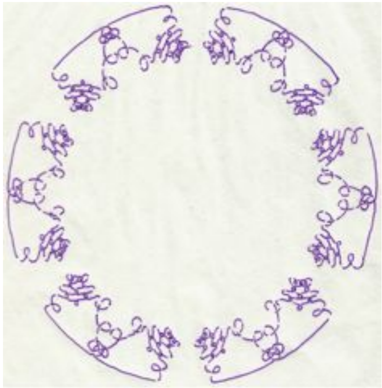

Figure 56. **Six copies of one of the outer components Link 4**

Figure 57. **Six copies of another of the outer components in Link 4**

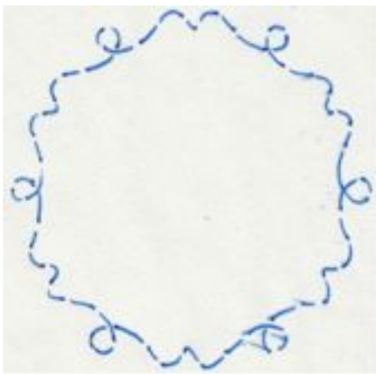

Figure 58. **One of the inner components Link 4**

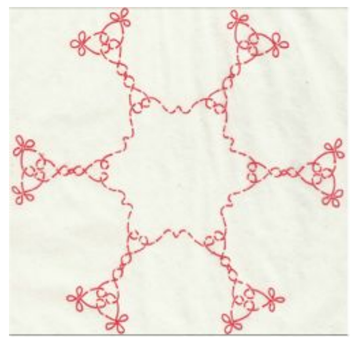

Figure 59. **Another of the inner components in Link 4**

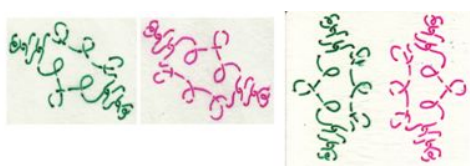

Figure 60. **Four orientations of one of the middle components of Link 4**

There is a core component repeated 24 times in four different orientations (Figure 60). Two collections of six of these components have the $D_6$ symmetry (Figures 61 and 62) while two other collections (Figures 63 and 64) have a specific 'pinwheel" rotation orientation thereby limiting their individual symmetry to the rotational group of order 6, $Z_6$. Taken together, however, we see their union once again achieves a $D_6$ symmetry group as the three dimensional rotation takes the 'right" facing orientation to the 'left" facing orientation.

There are 6 identical copies of each outer ring component and each copy is isolated from the other five of its kind. The large outer components (Figure 56) are each a connected sum of an $8_{18}$ knot and two copies of a knot we identify as a $12a690$ using the Hoste—Thistlethwaite enumeration. Each component has a $Z_2$ three dimensional rotational symmetry. The smaller outer components (Figure 57) are copies of this same twelve crossing alternating knot, and have the same $Z_2$ symmetry. The main body component (Figure 59) is the connected sum of six identical unknotted sub-components. The component as a whole has an evident hexagonal structure, and is topologically unknotted. There is an interesting symmetry in this component that does not appear in the other links. The entire component exhibits 6-fold symmetry, as evidenced by the hexagonal structure and the $D_6$ symmetry of the entire medallion. However, on each point of the hexagon there is 3-fold symmetry in the local triangular structure that consists of twisted quatrefoils that introduce an appearance of an intrinsic 4-fold symmetry.

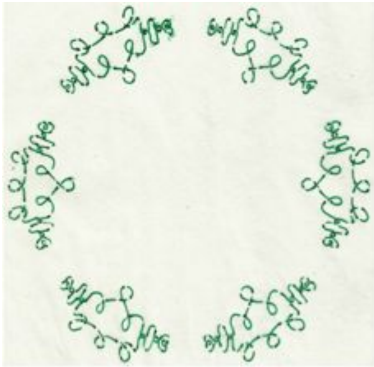

Figure 61. **One of the middle collections of components of Link 4 with $D_6$ symmetry**

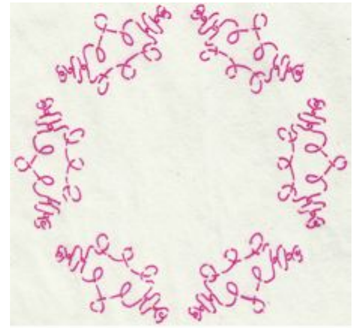

Figure 62. **Another of the middle collections of components in Link 4 with $D_6$ symmetry**

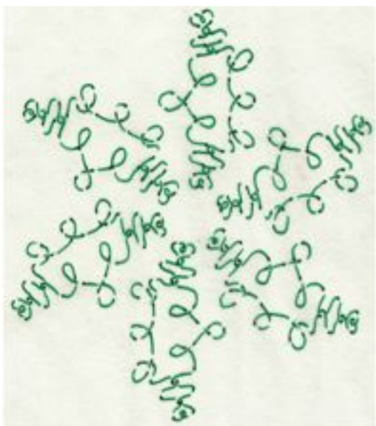

Figure 63. **One of the middle collections of components of Link 4 with $Z_6$ symmetry**

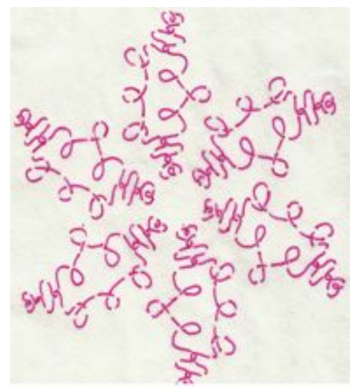

Figure 64. **Another of the middle collections of components in Link 4 with $Z_6$ symmetry**

The 24 copies of the fundamental component (Figure 60) are each topologically unknotted. There are 12 around the outer ring of the design—six components with the concave side facing the center of the circle (Figure 61), and six with the concave side facing away from the center (Figure 62). The remaining 12 copies are arranged around the center of the medallion in a pinwheel formation—six with the concave side directed clockwise (Figure 63), and six with the concave side directed counter clockwise (Figure 64). The inner ring is topologically unknotted and consists of six removable curls, preserving the $D_6$ symmetry (Figure 58).

Recall that Link 4 was classified as exhibiting the circular structure. When viewing the emblem, there are seven circles that dominate the impression of the design—one in the center and six around the outside. While the center circle and outer circles are locally identical, they connect to the rest of the medallion differently and are constructed in different manners. The six outer circles (Figure 65) are each comprised of 9 individual components: one $12a690$ component from the outer edge, two connected sum components from the outer edge, two pinwheel components (one in each direction), and four of the horizontal components (two in each direction). While the outer circle design itself is symmetric, the way in which it is constructed is not symmetric. In contrast, the center circle (Figure 66) is symmetric and exhibits $D_6$ symmetry. It is constructed with 12 individual pinwheel components (6 in each direction).

## 4.5 *Analysis of Link 5*

Link 5 (Figure 67) is similar to Link 4 in that it has a large number of components (22 in total as seen in Figure 68), but it differs from all the other links in that it is the only one to have $D_8$ symmetry.

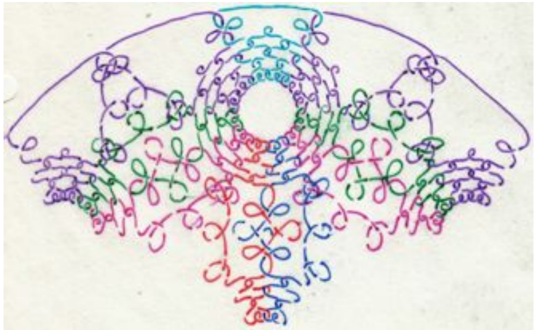

Figure 65. **One of the six outer circles Link 4**

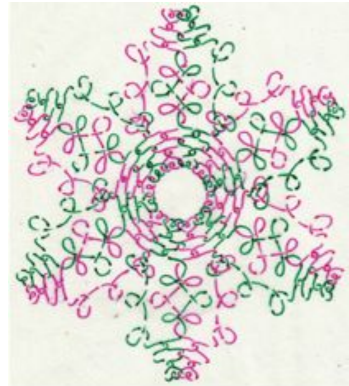

Figure 66. **The inner circle in Link 4 with $D_6$ symmetry**

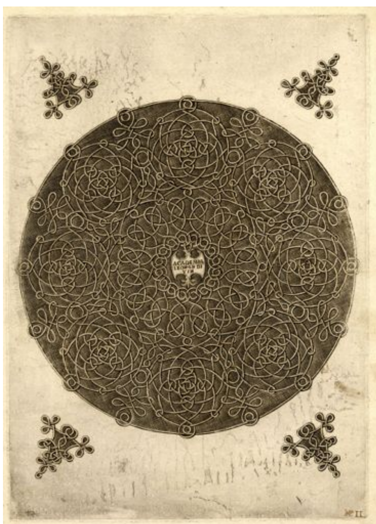

Figure 67. **Link 5 (With permission of the British Museum)**

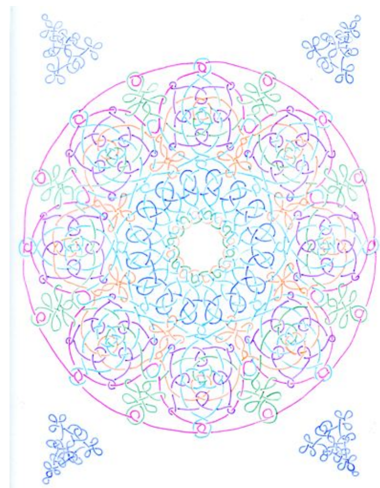

Figure 68. **Link 5 has six components and $D_8$ symmetry**

Figure 69. **The corner of Link 5**

### *4.5.1 The corner configurations*

The Link 5 corner knot (Figure 69) has 39 crossings, 13 of which are nugatory. Their removal gives a prime alternating 26 crossing knot. Its DT code is 4 12 14 32 28 16 2 6 22 50 46 36 40 10 34 8 30 42 24 26 38 48 20 52 18 44 and its Jones polynomial is $-6\ 20 : -1\ \ 8\ -38\ \ 133\ -372\ \ 872\ -1767\ \ 3163\ -5076\ \ 7389\ -9829\ \ 12020$ $-13562\ 14149\ -13663\ 12205\ -10068\ 7644\ -5315\ 3360\ -1911\ \ 963\ -420\ \ 153\ -44\ \ 9\ -1$. The core 26 crossing prime alternating knot is distinct from the 26 crossing prime alternating knot found in the corner of Link 3 as shown by the difference in their Jones polynomials.

Figure 70. **An outer component in Link 5**

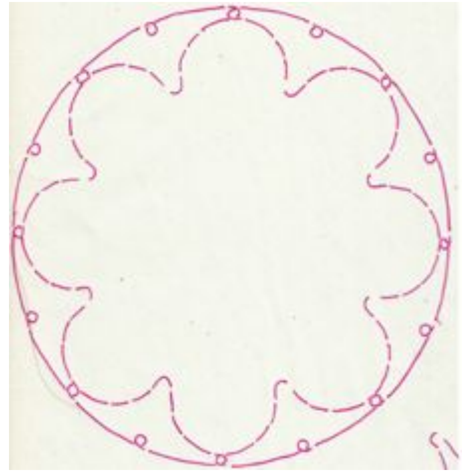

Figure 71. **The linking of eight of these outer components**

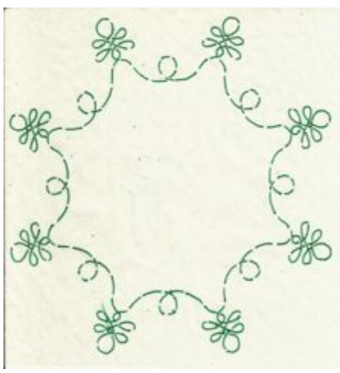

Figure 72. **Core component of Link 5 with $D_8$ symmetry**

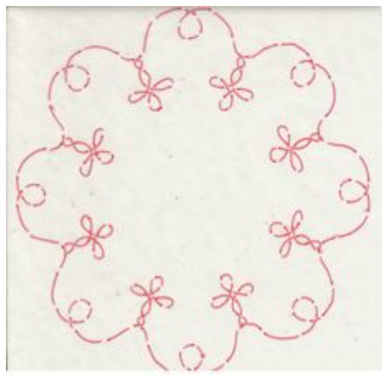

Figure 73. **Core component of Link 5 with $D_8$ symmetry**

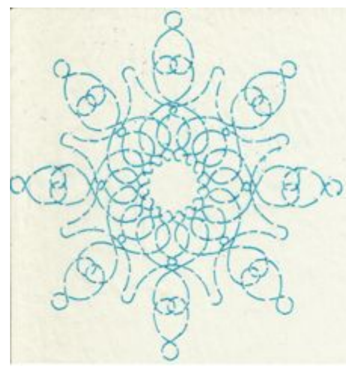

Figure 74. **Core component of Link 5 with $D_8$ symmetry**

### *4.5.2 The central link*

Exhibiting the $D_8$ symmetry group, Link 5 contains 22 components. There are eight interlinked copies of an outer component that form the perimeter (Figures 70 and 71). Each of these components has $Z_2$ symmetry and each copy differs from the previous one by a $\frac{\pi}{4}$ rotation that results in the global $D_8$ symmetry.

We refer to the three components that make up the core of the medallion as the outer ring, the ring of connected quatrefoils, and the main body component. The outer ring (Figure 72) is a topologically unknotted connected sum of eight copies of an unknot that we identify as the mirror image of the upper portion of the Link 3 corner pieces (Figure 43). Alternating with these knots are eight removable curls, ensuring the $D_8$ symmetry. The connected sum of quatrefoils (Figure 73) is also topologically unknotted and consists of eight removable curls that alternate with the twisted quatrefoils. The main body component (Figure 74) traverses the entire link, linking with every other component of the medallion. After removing the eight nugatory crossings we discover that this component is the connected sum of eight figure eight knots (Figure 17). However, even after removing the topologically irrelevant curls and the connected summands, what remains is a complex knot with an underlying structure unlike the previous knots. It is a 128 crossing, non-alternating knot.

The small component repeated eight times around the outer portion of the link is a quatrefoil embellished with nugatory curls (Figure 75). Unlike the twisted quatrefoils (Figure 15) ubiquitous throughout the six links, this component resembles the traditional quatrefoil prevalent in Gothic architecture and Christian symbolism (Figure 14). Eight copies of this quatrefoil component, isolated from one another, form a ring around the medallion (Figures 76). Each individual

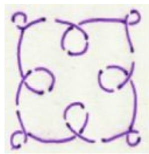

Figure 75. **Quatrefoil outer component**

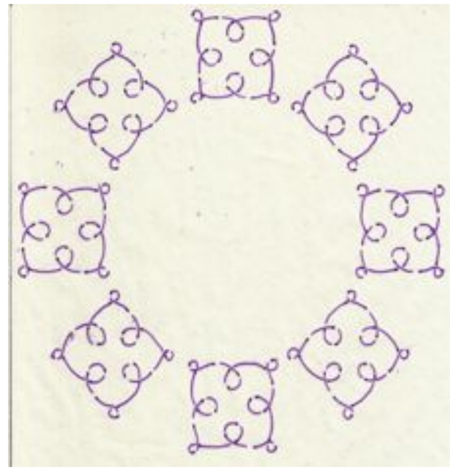

Figure 76. **The linking of eight of these outer components**

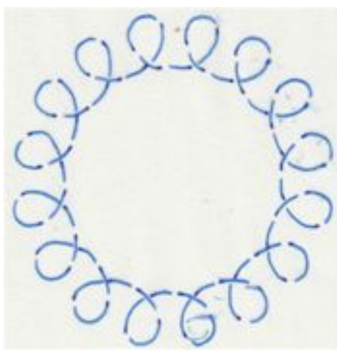

Figure 77. **Another middle component**

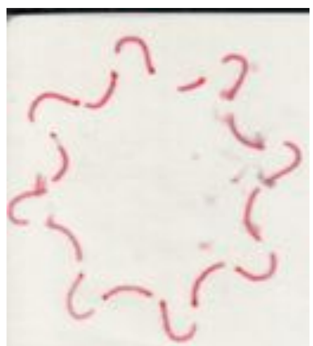

Figure 78. **An inner component**

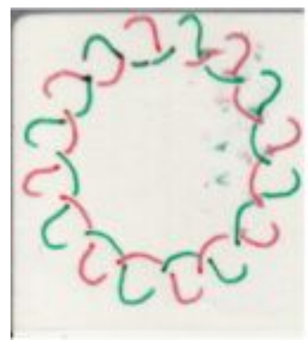

Figure 79. **Linking of the two inner components**

quatrefoil exhibits $D_4$ symmetry, and the eight copies of the component are arranged in a ring so as to preserve the global $D_8$ symmetry.

As classified in section 3, Link 5 exhibits both the radial and circular structures found among the cartelle designs. The three core components, outer ring, and eight isolated copies of the quatrefoil component form eight outer circles that together make a ring concentric with the central circle. The perimeter of the central circle is defined by the middle ring component—a topologically unknotted ring that consists of 16 nugatory curls (Figure 77). The final two inner ring components are identical copies of one another, differing only by a $\frac{\pi}{16}$ rotation (Figure 78). Each one is unknotted and contains no twists or curls. When both inner ring components are linked together the $D_{16}$ symmetry is preserved with a subgroup with a subgroup corresponding to the global $D_8$ symmetry (Figure 79).

### 4.6 *Analysis of Link 6*

Link 6, Figures 80 and 81, is comprised of 18 components and, like Link 4, has a $D_6$ symmetry group.

#### *4.6.1 The corner configurations*

The Link 6 corner (Figure 82) is a 32 crossing alternating knot which, when the 14 visible nugatory crossings are removed, gives a prime alternating 18 crossing knot. Its DT code is 4 10 22 18 2 30 32 26 8 24 6 20 28 16 36 34 12 14 and its Jones polynomial is $-17\ 1:\ \ 1\ -6\ \ 20\ -50\ \ 100\ -168\ \ 248\ -324\ \ 376\ -395\ \ 373\ -316\ \ 241\ -161\ \ 94\ -46\ \ 18\ -5\ \ 1.$

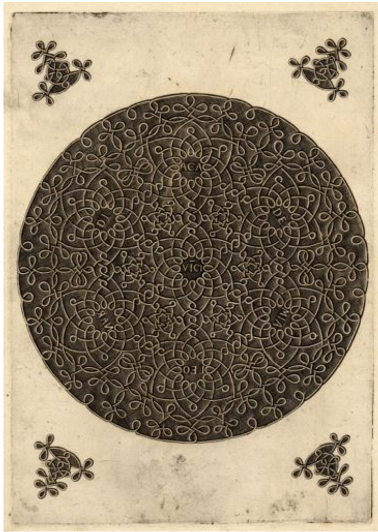

Figure 80. **Link 6 (With permission of the British Museum)**

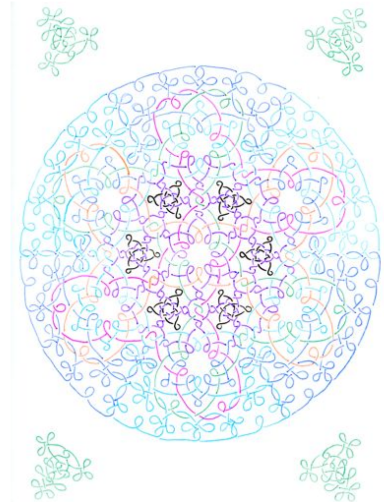

Figure 81. **Link 6 has 18 components and $D_6$ symmetry**

Figure 82. **The corner of Link 6**

#### *4.6.2 The central link*

An outer component of Link 6 has $D_3$ symmetry (Figure 83). There are two copies of this component differing from one another by a $\frac{\pi}{3}$ rotation. When linked together, these outer components preserve the global $D_6$ symmetry (Figure 84). Each of these outer components are connected sums of three identical pretzel knots that we identify as $p(3,3,3,3,3)$ (Figure 92). A pretzel knot is 'a knot obtained from a tangle represented by a finite number of integers separated by commas" [6], where a tangle is a region in the projection plane surrounded by a circle such that the knot or link crosses the circle exactly four times. In each pretzel knot, there appears to be three twisted quatrefoils and two pieces that resemble the twisted quatrefoils but are skewed and have a somewhat different structure. These pseudo quatrefoils (Figure 93) are anomalous cases in contrast to the regularity and symmetry of the knots and links seen thus far.

Three identical unknotted middle components, shaped like dog bones, each possess $D_2$ symmetry (Figure 85). Each one differs from its adjacent copy by a $\frac{\pi}{4}$ rotation, and the collection of the three copies maintains the aggregate symmetry (Figure 86). The linking of two adjacent middle components is 4, and the linking of all three is 12.

The core of the medallion is composed of six copies of the knot $9_{41}$, each containing an embedded trefoil. The $9_{41}$ (Figure 87) and trefoil (Figure 88) have $D_3$ symmetry and fit together to form a unit (Figure 89). The combination of the six copies creates an aggregate design with $D_6$ symmetry (Figure 90). The linking of each $9_{41}$-trefoil pairs is 6, related perhaps to the $D_6$ symmetry of the design.

The final component is an inner unknotted ring that exhibits the global $D_6$ symmetry group (Figure 91).

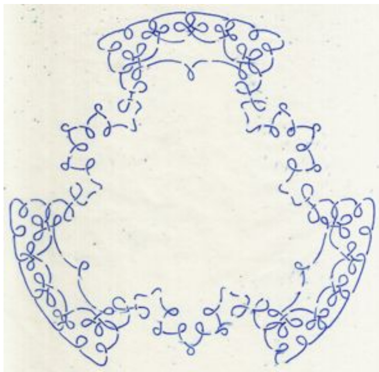

Figure 83. **An outer component of Link 6**

Figure 84. **The union of two outer components**

Figure 85. **A middle component of Link 6**

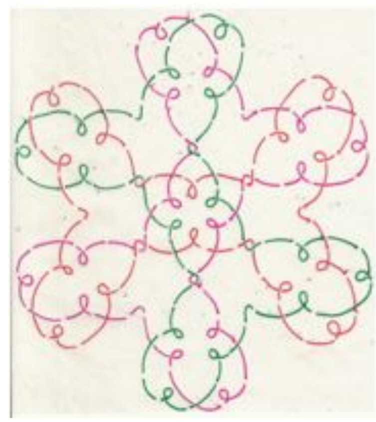

Figure 86. **The union of three middle components**

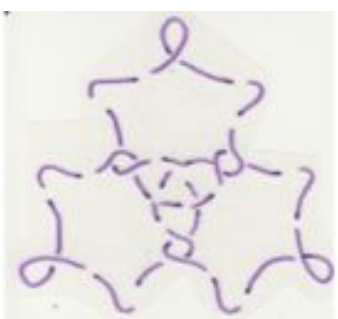

Figure 87. **A $9_{41}$ component**

Figure 88. **An inner component**

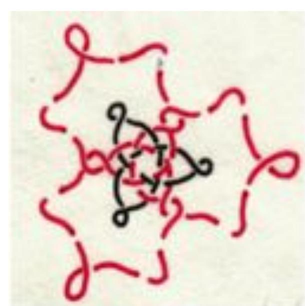

Figure 89. **Linking of the two inner components**

## 5. Summary discussion of the six links and corners

The symmetry groups observed in the structure of the links are, fundamentally, of two basic types: those naturally constructed from the square (reflected in the powers of two: 2, 4, 8, 16) and those naturally constructed from the triangle (three times the powers of two: 3, 6, 12). The straight edge and compass construction of both of these basic regular polygons was known in antiquity. The constructible regular polygons are the skeleton from which geometric symmetry is often based. The number of sides for which one can do this is any power of 2 , e.g. 4, 8, 16, 32 (all observed in Leonardo's cartelle), or a product of a power of two (including one) with a Fermat Prime. The known Fermat Primes are 3, 5, 17, 257, and 65537. Thus, the hexagonal symmetry is also one of the classically known natural symmetries. Contemporaries of Leonardo

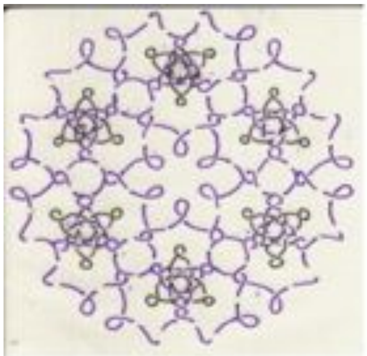

Figure 90. **The union of the $9_{41}$ and $3_1$'s in Link 6**

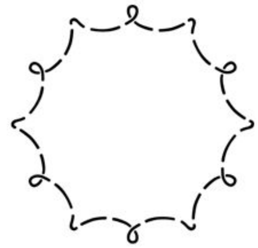

Figure 91. **The central component of Link 6**

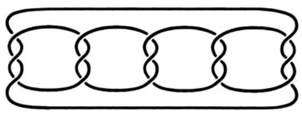

Figure 92. **A pretzel knot in an outer component in Link 6**

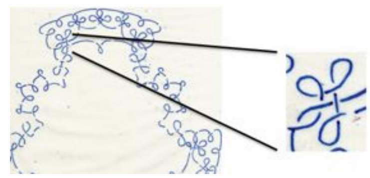

Figure 93. **An anomalous conformation in Link 6**

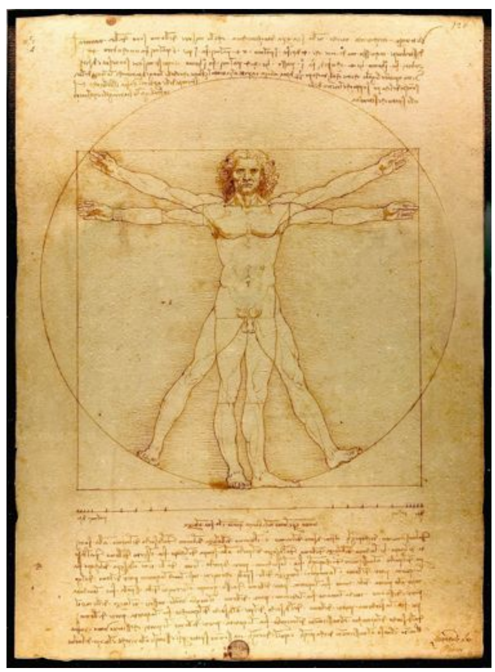

Figure 94. **Le Vitruvian Man, *Luc Viatour / www.Lucnix.be***

and, one presumes, Leonardo knew how to closely approximate the regular polygons for which an exact construction is now known to be impossible (7, 9, 11, 13, etc). Leonardo's symmetries, however, were constrained to the powers of two and three times a power of two. Given the degree of complexity of the cartelle and the proposition that they were to represent puzzles, one must wonder why more complex symmetries were not employed in this task. For example, does one encounter any instances of pentagonal, $D_5$, symmetry in Leonardo's other work? Some suggest that it is present in Le Vitruvian Man (Figure 94), but this contention seems only weakly supported by the actual drawing. One must wonder why $D_5$ is absent. Were there religious or contemporary cultural matters that caused this symmetry to be avoided?

When analyzing these entangled designs, one cannot help but wonder about the objectives of each, and the methods employed in their construction. The apparent overall regularity and consistency might suggest the possibility that they were not hand drawn, but created using a template. To test this hypothesis we employed geometry to measure the length scales and curvature of the seemingly identical core components, for example in Link 4 (Figures 61—64), and discovered significant variation. This suggests that the designs were indeed created by hand rather than a pattern. The intertwining components themselves could have been drawn using a method similar to that of the Pictish form of Celtic art. Another possibility is that a symmetric projection was first designed, and then later made to be alternating in the realization of the etching.

## 5.1 *Anomalies observed in the links*

Although the six figures exhibit consistency and symmetry, we have identified a few irregularities. The most striking of these concerns the corner configurations of Link 2. Here one finds a two component link of a trefoil and an unknot embellished with twisted quatrefoils. It is in the trefoils of these four corner knots that we find one inconsistency among the cartelle. The link in the lower right corner has a right-handed trefoil (all crossings are positive) whereas the other three have left-handed trefoils (all crossings are negative). One might wonder, was this difference in structure present in the original creation or did it arise in the course of copying the design? A similar irregularity exists in the corner knots of the Link 3 conformation. This ancillary figure is a single component, and consists topologically of two trefoils and a core complex prime knot. The two trefoils of the lower left corner knot are positive, whereas the trefoils of the other three are negative. Again, it is intriguing to question, how did this structural difference arise? Yet another irregularity exists in the corner knots of the Link 5 conformation. In this case, the ancillary figure is a single component that consists of a trefoil embedded in a core 26 crossing prime knot. The upper left and lower right components have positive trefoils, and the upper right and lower left components have negative trefoils. This instance of inconsistency is more symmetric than the other two cases given the pairing of the corner knots. Nevertheless, the question as to the origin of these differences presents itself, and especially in this case, we must wonder if there be some as yet not understood objective leading to their presence. Perhaps the corner conformations were produced by another artist, one who did not appreciate the distinction between the positive and negative trefoils. Or, perhaps, the distinction between positive and negative trefoils was one that was not appreciated by Leonardo.

With respect to the principal links, the situation is quite different. As far as we know, there is only one instance of an error. In Link 5, there is a point at which the curve is interrupted and a segment is missing (Figure 95). In the center of the circular conformation lying in the third quadrant of the medallion, the main body component (Figure 75) should cross over itself in order to maintain the regularity of the entire link, but instead it stops. There are thus two under crossings in a row, with a discontinuity in the curve between. This anomalous occurrence suggests the possibility of an error on the part of an apprentice or individual replicating the initial knotted design. One wonders whether or not all renditions or copies of this link have the same absent crossing.

In general, we believe that it is quite remarkable that the six woodcuts of Düred are faithful to the originals, even to the extent of accurately reproducing the trefoil anomalies. In the specific case of the missing segment, Dürear's woodcut has compressed the conformation at the location in question so as to obscure the structure a bit. Nevertheless, the image suggests fidelity there as well.

Such fidelity is not, however, the case with other instances of copies of Leonardo's art. For example, the detail on the Mona Lisa bodice, Figure 2 shows several irregular facets. Looking closely at the sequence of curls and quatrefoils, one notices that there is a conspicuous irregularity in the structure of the curls, changing without order between positive and negative curls. Furthermore, the number of curls between quatrefoils changes from two to three and back. When

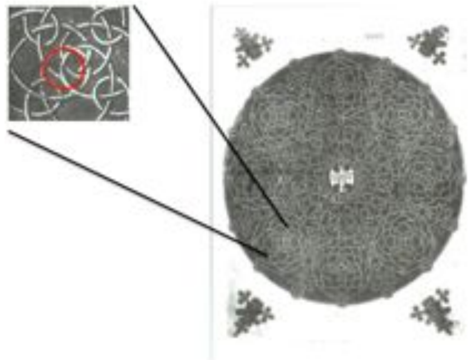

Figure 95. **The missing segment in Link 5**

Figure 96. **Prado Mona Lisa bodice [26]**

compared to the bodice of a recently restored contemporary copy of the Mona Lisa found in the Prado Museum, Figure 96, or the 'earlier' Mona Lisa (also known as the Ilesworth Mona Lisa), we see instances in which Leonardo's variation in structure is replaced by a periodic structure as well as striking differences in the structure of the quatrefoils. Specifically, in the chain of curls, the Mona Lisa has a sequence of three curls whereas the others are all sequences of two and the quatrefoils are alternating. In the Prado and Ilesworth copies, this is not the case, all are sequences of double curls. In each of the instances, the quatrefoils display different variations in the crossing structures. The Mona Lisa Foundation's website reports some of these elements, but not all, in their comparison: http://monalisa.org/2012/09/08/the-embroidery/. For example, Leonardo's curls change sign as do those in the Ilesworth copy, while those in the Prado copy do not. Unlike Dürer, it is curious that these 'copies" have some striking variations in topological character.

## 6. Conclusions

It is thought that the cartelle designs were created for the 'Academia Leonardi Vinci' as puzzles—that is, for pure entertainment. They are clearly quite puzzling, reflecting a substantial variation in structure as reflected in the groups of symmetry associated with each link and its constituent pieces, and significant variation in the number of components and in the quantitative nature of the knotting and linking. Even the artistic general appearance of the links reflect a variety of inspiration for the medallions. Given the historical context of their creation one can also speculate that Leonardo made use of familiar symbols and designs that, combined with the desire for certain symmetries, resulted in these complex knotted designs. Exactly how this was accomplished is not clear from our analysis though we do argue that, as a consequence of the local geometric variation and the presence of trefoil anomalies, much of the work must have been accomplished 'by hand.' The absence of pentagonal symmetry raises, we believe, an interesting question. Is this a consequence of being unaware of how to make such symmetries or are there cultural forces that caused the artist to avoid such symmetry?

## 7. Acknowledgements

We thank Caroline Cocciardi for introducing KCM to the knots in Leonardo's paintings and in the cartelle. Her enthusiastic research into the subject of knots in Leonardo's work was the inspiration for our UCSB honors seminar and all that followed. KCM is grateful for the dedicated exploration of Leonardo's cartelle and the underlying mathematics by the six honors students, Corrine Cooper, Jessica Hoy, David Hyde, Bryan Lee, and Erica Mason which was the catalyst for this project. We thank Corrine Cooper for sharing the results of her investigation of the history associated with the cartelle. Jill Pederson's 2008 Ph.D. dissertation at Johns Hopkins Universtiy was an especially helpful source in guiding our understanding of the history of the cartelle. JH provided the line drawings of the cartelle and their components that formed the foundation of much of our analysis. We are grateful to the British Museum for permission to use Link 3, 5, &6. Finally, we wish to acknowledge the contribution of Ethan Turpin who provided us with excellent line drawings that appear in some of the critical figures in this paper and thank him for this contribution that helped us complete this study.

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