Mathematics 108A: Practice Quiz D
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Recall the following Lemma which was proven in the class earlier:

**Linear Dependence Lemma.** Suppose that \((v_1, \ldots, v_m)\) is a linearly dependent list of vectors in a vector space \(V\) over a field \(F\), and that \(v_1 \neq 0\). Then there exists \(j \in \{2, \ldots, m\}\) such that

\[
v_j \in \text{Span}(v_1, \ldots, v_{j-1}).
\]

Moreover,

\[
\text{Span}(v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_m) = \text{Span}(v_1, \ldots, v_m).
\]

Assuming this lemma, we would like to prove the MAIN RESULT of Chapter 1 in the text:

**Replacement Theorem.** If \(V\) is a vector space over a field \(F\), \((u_1, \ldots, u_m)\) is a linearly independent list of elements of \(V\) and \(V\) is the span of a list \((w_1, \ldots, w_n)\), then \(m \leq n\).

Idea of proof: One by one replace elements of the spanning list by elements of the linear independent list, renormalizing to the same size by means of the Linear Dependence Lemma.

1. Carry out the proof of step 1: Let \(B_0 = (w_1, \ldots, w_n)\). Show that we can replace one of the elements of \(B_0\) by \(u_1\), thereby obtaining a list \(B_1\) of length \(n\) which spans \(V\) and contains \(u_1\).

Since \(B_0\) spans \(V\), \(\vec{w}_1 \in \text{Span}(\vec{w}_1, \ldots, \vec{w}_n)\) and \((\vec{u}, \vec{w}_1, \ldots, \vec{w}_n)\) is linearly dependent. \(\vec{u} \neq \vec{0}\) so the Linear Dependence Lemma implies \(\exists \vec{w}_j \in (\vec{w}_1, \ldots, \vec{w}_n)\) such that \(\vec{w}_j \in \text{Span}(\vec{u}, \vec{w}_1, \ldots, \vec{w}_{j-1})\)

and \(\text{Span}(\vec{u}, \vec{w}_1, \ldots, \vec{w}_{j-1}, \vec{w}_j, \ldots, \vec{w}_n) = \text{Span} B_0 = V\).

Let \(B_1 = (\vec{u}, \vec{w}_1, \ldots, \vec{w}_{j-1}, \vec{w}_{j+1}, \ldots, \vec{w}_n)\)
2. Carry out the proof of step \( j \) for \( 2 \leq j \leq m \): Suppose that \( B_{j-1} \) has length \( n \), spans \( V \) and starts with \( (u_1, \ldots, u_{j-1}) \). Show that we can replace one of the elements of \( B_{j-1} \) by \( u_j \), thereby obtaining a list \( B_j \) of length \( n \) which spans \( V \) and contains \( (u_1, \ldots, u_j) \).

\[
B_{j-1} = (\vec{u}_1, \ldots, \vec{u}_{j-1}, \vec{w}_{\alpha(j)}, \ldots, \vec{w}_{\alpha(n)}) \text{ spans } V.
\]

Hence \( \vec{u}_j \in \text{Span} (\vec{u}_1, \ldots, \vec{u}_{j-1}, \vec{w}_{\alpha(j)}, \ldots, \vec{w}_{\alpha(n)}) \) and \((\vec{u}_1, \ldots, \vec{u}_j, \vec{w}_{\alpha(j)}, \ldots, \vec{w}_{\alpha(n)})\) is linearly dependent \( \vec{u}_1 \neq \vec{0} \) and \( \vec{u}_k \notin \text{Span} (\vec{u}_1, \ldots, \vec{u}_{k-1}) \) for \( 1 \leq k \leq j \).

Hence the Linear Dependence Lemma implies \( \exists \vec{w}_{\alpha(k)} \), \( j \leq k \leq n \), such that \( \vec{w}_{\alpha(k)} \in \text{Span} (\vec{u}_1, \ldots, \vec{u}_j, \vec{w}_{\alpha(j)}, \ldots, \vec{w}_{\alpha(k-1)}) \) and \( \text{Span} (\vec{u}_1, \ldots, \vec{u}_j, \vec{w}_{\alpha(j)}, \ldots, \vec{w}_{\alpha(k-1)}, \vec{w}_{\alpha(k)}, \ldots, \vec{w}_{\alpha(n)}) = \text{Span} B_{j-1} = V \).

Let \( B_j = (\vec{u}_1, \ldots, \vec{u}_j, \vec{w}_{\alpha(j)}, \ldots, \vec{w}_{\alpha(k-1)}, \vec{w}_{\alpha(k+1)}, \ldots, \vec{w}_{\alpha(n)}) \).

After step \( m \), we have added all elements of \((u_1, \ldots, u_m)\) and the process stops. The final list has \( n \) elements and contains \((u_1, \ldots, u_m)\), so \( m \leq n \). QED

3. Now use the Replacement Theorem to prove the following corollary:

**Corollary.** If \( V \) is a vector space over a field \( F \), and \( (u_1, \ldots, u_m) \) and \( (w_1, \ldots, w_n) \) are two bases of \( V \), then \( m = n \).

\[
\begin{align*}
(u_1, \ldots, u_m) \text{ linearly independent} \\
\text{and } (w_1, \ldots, w_n) \text{ span } V & \quad \Rightarrow \quad m \leq n \\
(w_1, \ldots, w_n) \text{ linearly independent} \\
\text{and } (u_1, \ldots, u_m) \text{ span } V & \quad \Rightarrow \quad n \leq m
\end{align*}
\]