One of the Main Theorems from Chapter 2 of the text is:

**Theorem.** If $V$ is a finite dimensional vector space and $T : V \rightarrow W$ is a linear map into a vector space $W$, then

$$\dim V = \dim N(T) + \dim R(T).$$

The proof of this theorem is based upon the theory of linear independence and bases presented in Chapter 1. We start by choosing a basis $(u_1, \ldots, u_m)$ for $N(T)$. The Extension Theorem states that we can extend this to a basis $(u_1, \ldots, u_m, w_1, \ldots, w_n)$ of $V$. If we can show that $(T(w_1), \ldots, T(w_n))$ is a basis for $R(T)$, then $\dim R(T) = n$. It will then follow that

$$\dim V = m + n = \dim N(T) + \dim R(T),$$

and the theorem will be proven. Thus we need only carry out the following two steps:

1. Prove that the list $(T(w_1), \ldots, T(w_n))$ spans $R(T)$.
2. Prove that the list $(T(w_1), \ldots, T(w_n))$ is linearly independent.

**Solution:** Suppose that $w \in R(T)$. Then there exists $v \in V$ such that $T(v) = w$. We can write

$$v = a_1 u_1 + \cdots + a_m u_m + b_1 w_1 + \cdots + b_n w_n,$$

and then

$$T(v) = a_1 T(u_1) + \cdots + a_m T(u_m) + b_1 T(w_1) + \cdots + b_n T(w_n),$$

and since $u_1, \ldots, u_m$ lie in null($T$),

$$w = T(v) = b_1 T(w_1) + \cdots + b_n T(w_n).$$

Thus $w$ lies in the span of $(T(w_1), \ldots, T(w_n))$.

2. Prove that the list $(T(w_1), \ldots, T(w_n))$ is linearly independent.

**Solution:** Suppose that

$$b_1 T(w_1) + \cdots + b_n T(w_n) = 0,$$
for some elements $b_1, \ldots, b_n$ of the field. Then

$$T(b_1w_1 + \cdots + b_nw_n) = 0,$$

so $b_1w_1 + \cdots + b_nw_n \in N(T)$.

But then

$$b_1w_1 + \cdots + b_nw_n = a_1u_1 + \cdots + a_mu_m,$$

for some choice of scalars $a_1, \ldots, a_m$. Thus

$$a_1u_1 + \cdots + a_mu_m - b_1w_1 - \cdots - b_nw_n = 0.$$

Since the list $(u_1, \ldots, u_m, w_1, \ldots, w_n)$ is linearly independent,

$$a_1 = \cdots = a_m = b_1 = \cdots = b_n = 0.$$

From this we conclude that $(T(w_1), \ldots, T(w_n))$ is linearly independent.