Self-intersections of Closed Parametrized Minimal Surfaces in Generic Riemannian Manifolds

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Abstract

This article shows that for a generic choice of Riemannian metric on a compact manifold M of dimension at least five, all prime compact parametrized minimal surfaces within M are imbeddings. Moreover, if M has dimension four, all prime compact parametrized minimal surfaces within M have transversal self-interstions, and at any self-intersection the tangent planes fail to be complex for any choice of orthogonal complex structure in the tangent space.

1 Introduction

This article presents an application of the bumpy metric theorem for compact parametrized minimal surfaces in Riemannian manifolds which was established in an earlier article [4]. We will show that if M has dimension at least four, then for generic choice of Riemannian metric on M, all prime compact parametrized minimal surfaces in M have transversal crossings in general position.

We say that a parametrized minimal surface $f : \Sigma \to M$ is *prime* if it is nonconstant and is not a nontrivial cover (possibly branched) of another parametrized minimal surface $f_0 : \Sigma_0 \to M$ of lower energy, where Σ_0 may be nonorientable. By a *generic choice of Riemannian metric* on M we mean a metric belonging to a countable intersection of open dense subsets of the spaces of L^2_k Riemannian metrics on M, as k ranges over the positive integers.

If $\mathrm{Map}(\Sigma, M)$ is the space of smooth maps from a surface Σ of genus g to M and $\mathcal{T}$ is the Teichmüller space of marked conformal structures on compact connected surfaces of genus g, then (as explained in [4]) a parametrized minimal surface $f : \Sigma \to M$ can be regarded as a critical point for the energy function

$$E : \mathrm{Map}(\Sigma, M) \times \mathcal{T} \to \mathbb{R}, \quad \text{defined by} \quad E(f, \omega) = \frac{1}{2}\int_\Sigma |df|^2 dA.$$

In this formula, $|df|$ and dA are calculated with respect to some Riemannian metric on Σ which lies within the conformal class $\omega \in \mathcal{T}$. The energy function E is invariant under an action of the mapping class group Γ on the product space $\mathrm{Map}(\Sigma, M) \times \mathcal{T}$, and thus descends to a map on the quotient

$$E : \mathcal{M}(\Sigma, M) \to \mathbb{R}, \quad \text{where} \quad \mathcal{M}(\Sigma, M) = \frac{\mathrm{Map}(\Sigma, M) \times \mathcal{T}}{\Gamma}, \tag{1}$$

a space which projects to the moduli space $\mathcal{R} = \mathcal{T}/\Gamma$ of conformal structures on Σ.

The bumpy metric theorem of [4] states that for generic choice of Riemannian metric on a manifold M of dimension at least four, all prime compact oriented parametrized minimal surfaces $f : \Sigma \to M$ are free of branch points and are as nondegenerate (in the sense of Morse theory) as allowed by the group G of conformal automorphisms of Σ which are homotopic to the identity. If G is the trivial group, they are Morse nondegenerate in the usual sense, while if G has positive dimension, they lie on nondegenerate critical submanifolds which have the same dimension as G. (By a *nondegenerate critical submanifold* for $F : \mathcal{M} \to \mathbb{R}$, where $\mathcal{M}$ is a Banach manifold, we mean a submanfold $S \subset \mathcal{M}$ consisting entirely of critical points for F such that the tangent space to S at a given critical point is the space of Jacobi fields for F.) The group G has positive dimension when the orientable surface Σ has genus one or zero, in which case G is $S^1 \times S^1$ or $PSL(2, \mathbb{C})$, respectively. A corresponding bumpy metric theorem can also be proven for nonorientable surfaces by using orientable double covers, the basic ideas being described in §11 of [4].

Our first goal is to use the bumpy metric theorem to show that when the dimension of the ambient manifold M is at least four, compact parametrized minimal surfaces in M are generically immersions with transversal crossings. We can define this notion (in accordance with [2] , Chapter III, §3) by considering the following subset of the s-fold cartesian product Σ^s for s a positive integer,

$$\Sigma^{(s)} = \{(p_1, \dots p_s) \in \Sigma^s : p_i \neq p_j \text{ when } i \neq j\},$$

as well as the multidiagonal in the s-fold cartesian product M^s,

$$\Delta_s = \{(q_1, \dots q_s) \in M^s : q_1 = q_2 = \cdots = q_s\}.$$

We then say that an immersion $f : \Sigma \to M$ has *transversal crossings* if for every $s > 1$, the restriction of

$$f^s = f \times \cdots \times f : \Sigma^s \longrightarrow M^s$$

to $\Sigma^{(s)}$ is transversal to Δ_s. (This is called an immersion with normal crossings in [2], but we avoid this terminology since this conflicts with the usual usage of the term normal within Riemannian geometry.) Thus if Σ is a compact surface and M has dimension at least five, an immersion with transversal crossings is a one-to-one immersion and hence an imbedding, while if M has dimension four, such an immersion has only double points and the intersections at double points are transverse.

Theorem 1. *Suppose that M is a compact connected manifold of dimension at least four. Then for a generic choice of Riemannian metric on M,*

1. *every prime compact parametrized minimal surface $f : \Sigma \to M$ is an immersion with transversal crossings,*
2. *any two distinct prime compact parametrized minimal surfaces have transverse intersections, and*
3. *if M has dimension four, then at any self-intersection point, the tangent planes are in general position with respect to the metric, that is, they are not simultaneously complex for any orthogonal complex structure on the tangent space.*

We emphasize that the minimal surfaces considered in Theorem 1 are not required to be area-minimizing or even stable.

Recall that according to one of the well-known theorems of Sacks and Uhlenbeck [6], a set of generators for the second homology group $H_2(M;\mathbb{Z})$ of a compact simply connected smooth Riemannian manifold M can be represented by area minimizing minimal two-spheres. Theorem 2 shows that when the metric on M is generic, these generators are represented by imbedded minimal two-spheres, so long as the ambient manifold has dimension at least five, and by immersions with at worst transverse double points when the ambient manifold has dimension four. When M has dimension four, the generic condition on the tangent planes enables us to use a result of Frank Morgan [5] to show that if $f : \Sigma \to M$ is a surface of genus g which minimizes area in some homology class, and f has points of self-intersection, then one of the self-intersections can be removed by surgery, producing a surface of larger genus and smaller area in the same homology class.

We say that an element $x \in H_2(M;\mathbb{Z})$ is *primitive* if it is not of the form my for some $y \in H_2(M;\mathbb{Z})$ and $m \geq 2$.

Theorem 2. *Suppose that M is a compact simply connected manifold of dimension at least four with a generic choice of Riemannian metric. Then each primitive element of $H_2(M;\mathbb{Z})$ is represented by an area-minimizing collection of disjoint imbedded parametrized minimal surfaces.*

Assuming Theorem 1, we can prove Theorem 2 as follows. Results of Almgren and Chang [1] (see the Main Regularity Result on page 72 of [1]) imply that any homology class is represented by an area minimizing integral current which arises from a smooth submanifold except for possible branch points and self-intersections. This can be represented by a finite collection of parametrized minimal surface $f_i : \Sigma_i \to M$, where each Σ_i is connected. When the metric is generic, it follows from the Main Theorem of [4] that each such f_i is free of branch points, while when the dimension of M is at least five, it follows from Theorem 1 that there are no self-intersections, or intersections between different components. When the dimension of M is four, Theorem 1 states that at the self-intersections the two tangent planes cannot be simultaneously complex for any complex structure. It therefore follows from Theorem 2 of [5] that if any f_i has nontrivial self-intersections, one of the self-intersections could be eliminated with a decrease in area, thereby contradicting the fact that the current is area minimizing. Thus the f_i's must be imbeddings. Similarly, area could be decreased if the images of different f_i's were not mutually disjoint, again contradicting area minimization. This proves Theorem 2.

We remark that Theorem 2 is related to an earlier result of Brian White [9] for unoriented surfaces. We will give a proof of Theorem 1 in the next section.

2 Proof of Theorem 1

In order to apply the Sard-Smale Theorem [8], it is convenient to replace $\mathrm{Map}(\Sigma, M)$ and $\mathrm{Met}(M)$ of smooth maps and smooth Riemannian metrics on M by their Sobolev completions $L^2_k(\Sigma, M)$ and $\mathrm{Met}_{k-1}(M)$, for a large integer k, which are Banach manifolds rather than Fréchet manifolds. However, to keep the notation simple, we will continue to denote these completions by $\mathrm{Map}(\Sigma, M)$ and $\mathrm{Met}(M)$. (Here $\mathrm{Met}_{k-1}(M)$ denotes the L^2_{k-1} completion of the space of smooth Riemannian metrics on M.) It is shown in [4] that

$$\mathcal{P}_\emptyset = \{(f, \omega, g) \in \mathrm{Map}(\Sigma, M) \times \mathcal{T} \times \mathrm{Met}(M) : \\ f \text{ is a prime immersed conformal } \omega\text{-harmonic map } \}. \quad (2)$$

is a smooth submanifold. The Main Theorem of [4] implies that if g_0 is generic (or "bumpy") metric on M, then all prime conformal harmonic maps are immersed and hence lie in $\mathcal{P}_\emptyset$. Moreover, for any such metric, each element of

$$\mathcal{N}_{g_0} = \pi_2^{-1}(g_0) \cap \mathcal{P}_\emptyset$$

is either a nondegenerate critical point for the energy, or lies in a nondegenerate critical submanifold which has the same dimension as the group G of symmetries for Σ. Here

$$\pi_2 : \mathrm{Map}(\Sigma, M) \times \mathcal{T} \times \mathrm{Met}(M) \longrightarrow \mathrm{Met}(M)$$

is the projection on the last factor. Finally, it follows from Lemma 6.1 of [4] that if (f, ω, g_0) is any element of $\mathcal{N}_{g_0}$, then since the orbits of the G-action generate the tangential Jacobi fields, the projection on the first factor,

$$\pi_0 : \mathcal{P}_\emptyset \longrightarrow \mathrm{Map}(\Sigma, M) \quad \text{has surjective differential at } (f, \omega, g_0). \quad (3)$$

Thus all maps f' sufficiently close to f lie in the image of $\mathcal{P}_\emptyset$, and can be realized by parametrized minimal surfaces for metrics near g_0.

We now prove the first statement of Theorem 1. We will construct a countable cover of $\mathrm{Map}(\Sigma, M) \times \mathcal{T} \times \mathrm{Met}(M)$ by product open balls $U_i \times V_i$,

$$U_i \subset \mathrm{Map}(\Sigma, M) \times \mathcal{T}, \qquad V_i \subset \mathrm{Met}(M),$$

such that if $U_i \times V_i$ intersects $\mathcal{P}_\emptyset$,

1. it is the domain for a submanifold chart for $\mathcal{P}_\emptyset$,
2. the restriction of $\pi_2 : U_i \times V_i \to V_i$ to $\mathcal{P}_\emptyset \cap (U_i \times V_i)$ is proper, and
3. the restriction of $\pi_1 : U_i \times V_i \to U_i$ to $\mathcal{P}_\emptyset \cap (U_i \times V_i)$ is a submersion.

The second condition can be arranged by Theorem 1.6 of [8] and the last condition follows from (3).

It follows from standard transversality theory for finite-dimensional manifolds (see §2 of Chapter 3 of [3] or Proposition 3.2 of Chapter III, §3 of [2]) that

$$\mathcal{U} = \{ f \in L^2_k(\Sigma, M) : \ f \text{ has transversal crossings } \}$$

is an open dense subset of $L^2_k(\Sigma, M) = \mathrm{Map}(\Sigma, M)$. Since π_0 is a submersion, $\pi_0^{-1}(\mathcal{U}) \cap \mathcal{P}_\emptyset \cap (U_i \times V_i)$ is an open dense subset of $\mathcal{P}_\emptyset \cap (U_i \times V_i)$.

It follows that for g in an open dense subset V_i' of V_i, the immersions in

$$\mathcal{N}_g \cap (U_i \times V_i), \quad \text{where} \quad \mathcal{N}_g = \pi_2^{-1}(g) \cap \mathcal{P}_\emptyset$$

have transversal crossings. Note that $W_i = V_i' \cup (\mathrm{Met}(M) - \overline{V}_i)$ is open and dense. Metrics g which lie in the intersections of the W_i's, a countable intersection of open dense subsets of $\mathrm{Met}(M)$, have the property that $\mathcal{N}_g$ contains only immersions with transversal crossings. This finishes the proof of the first statement of Theorem 1.

To prove the second statement, we employ the same argument, modified to the case where Σ is a compact surface with two components instead of one.

To prove the last statement in the case where M has dimension four, we need to construct a variation of the metric which puts a given intersection into general position. The argument for the first statement of the Theorem shows that we need only consider one transversal intersection at a time.

Suppose that p and q are distinct points of Σ and that $f(p) = f(q)$ and let v_1 and V_2 be disjoint open neighborhoods of p and q in Σ. We construct coordinates (u^1, u^2, u^3, u^4) on a neighborhood U of $f(p)$ in M so that

1. $u^i(f(p)) = 0$,
2. $f(V_1) \cap U$ is described by the equations $u^3 = u^4 = 0$,
3. $f(V_2) \cap U$ is described by the equations $u^1 = u^2 = 0$,
4. $f^*\langle \cdot, \cdot \rangle | V_1 = \lambda_1^2((dx^1)^2 + (dx^1)^2)$, where $x^a = u^a \circ f$, and
5. $f^*\langle \cdot, \cdot \rangle | V_2 = \lambda_2^2((dx^3)^2 + (dx^4)^2)$, where $x^r = u^r \circ f$.

Let g_{ij} be the components of the metric in these coordinates, so that

$$g_{ab} = \lambda_1^2 \delta_{ab}, \qquad g_{rs} = \lambda_2^2 \delta_{rs}.$$

We assume that at the intersection point, $f_*(T_p\Sigma)$ and $f_*(T_q\Sigma)$ are simultaneously complex for some orthogonal complex structure on TM. (We can then assume without loss of generality that $g_{13} = g_{24}$ and $g_{14} = -g_{23}$.)

If we define the Christoffel symbols in terms of the metric

$$\Gamma_{k,ij} = \frac{1}{2}\left(\frac{\partial g_{ki}}{\partial u^i} + \frac{\partial g_{kj}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k}\right), \quad \Gamma^k_{ij} = \sum g^{kl}\Gamma_{l,ij},$$

the fact that f is harmonic is expressed by the equations

$$\Gamma^k_{11} + \Gamma^k_{22} = 0 \quad \text{along } f(V_1), \qquad \Gamma^k_{33} + \Gamma^k_{44} = 0 \quad \text{along } f(V_2). \tag{4}$$

We will construct a variation in the metric $(\dot{g}_{ij})$ such that $\dot{g}_{ab} = 0 = \dot{g}_{rs}$ and the equations (4) continue to hold. The resulting variation $\dot{\Gamma}_{k,ij}$ in the Christoffel symbols will then satisfy the equations

$$\dot{\Gamma}_{b,aa} = 0, \quad \dot{\Gamma}_{r,aa} = \frac{\partial \dot{g}_{ra}}{\partial u^a}, \quad \dot{\Gamma}_{s,rr} = 0, \quad \dot{\Gamma}_{a,rr} = \frac{\partial \dot{g}_{ra}}{\partial u^r}.$$

Thus we want to arrange that

$$\sum_a \frac{\partial \dot{g}_{ra}}{\partial u^a} = 0 \quad \text{along } f(V_1), \quad \text{and} \quad \sum_r \frac{\partial \dot{g}_{ra}}{\partial u^r} = 0 \quad \text{along } f(V_2). \tag{5}$$

If we construct a smooth function $h : U \to \mathbb{R}$ and then set

$$\begin{pmatrix} \dot{g}_{13} & \dot{g}_{14} \\ \dot{g}_{23} & \dot{g}_{24} \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 h}{\partial u^2 \partial u^4} & -\frac{\partial^2 h}{\partial u^2 \partial u^3} \\ -\frac{\partial^2 h}{\partial u^1 \partial u^4} & \frac{\partial^2 h}{\partial u^1 \partial u^3} \end{pmatrix},$$

we find that the equations (5) are satisfied. We can choose such a function which has compact support within U, and for which

$$\begin{pmatrix} \dot{g}_{13} & \dot{g}_{14} \\ \dot{g}_{23} & \dot{g}_{24} \end{pmatrix}(f(p))$$

is arbitrary. The resulting metric perturbation will preserve conformality and minimality of f as required, yet can be chosen so that after perturbation $f_*(T_p\Sigma)$ and $f_*(T_q\Sigma)$ will not be simultaneously complex for some orthogonal complex structure on $T_{f(p)}M = T_{f(q)}M$.

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