Math 3CI: Project 3
Foxes and rabbits

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We have seen how differential equations, or their discrete analogs, can be used to model infectious disease and population growth for a single species. We now consider a model for population growth for two species of animals, one eating the other. This leads to interesting models of interaction between the species. The ideas form the basis for modern population ecology as developed by Vito Volterra (1860-1940) and Alfred Lotka (1880-1949).

The first problem is to construct two equations that model the populations of foxes and rabbits on an island. Suppose that we let

\[ x(t) = \text{(population of rabbits at time } t) \], \quad y(t) = \text{(population of foxes at time } t) \].

We would like our equations to incorporate the following assumptions:

a. In the absence of foxes, the rabbit population grows at a rate proportional to the population. Thus in this case, we expect the rabbit population to satisfy the equation of exponential growth:

\[ \frac{dx}{dt} = Ax \]

where \( A \) is a constant (the instantaneous birth rate minus death rate).

b. When foxes are present, there is an additional contribution to the death rate due to rabbits being eaten by the foxes. This contribution should be proportional to the product \( xy \) of rabbits and foxes.

c. If no rabbits are present, the fox population should die off at a rate proportional to the number of foxes.

d. If rabbits are present, the birth rate of foxes should increase because of the available food supply. The contribution to the increase should be proportional to the product \( xy \) of rabbits and foxes.

Last time, we saw that this gives rise to a system of differential equations

\[ \frac{dx}{dt} = Ax - Bxy \]

\[ \frac{dy}{dt} = -Cy + Dxy \],

(1)
where we might assume that $A$, $B$, $C$ and $D$ are all positive constants. These are known as the Volterra-Lotka predator-prey equations. Another possible model for interaction would include an additional term in the first equation to account for overcrowding of the rabbits. This leads to the system

\[
\frac{dx}{dt} = Ax - Bxy - Ex^2
\]
\[
\frac{dy}{dt} = -Cy + Dxy,
\]

It is not actually possible to completely solve the systems (1) and (2) and get simple explicit answers in terms of familiar functions. In real life, one has to study such systems by qualitative methods and obtain numerical solutions.

1. The simplest solutions to (1) and (2) are the constant solutions. Find these.

2. For system (1), find the portions of the $(x, y)$-plane for which

\[
\frac{dx}{dt} > 0 \& \frac{dy}{dt} > 0, \quad \frac{dx}{dt} > 0 \& \frac{dy}{dt} < 0, \quad \text{and so forth.}
\]

What does this information suggest about the qualitative behavior of the solutions?

3. To make the system (1) more specific, let us assume that time is measured in months and

\[
A = .2, \quad B = .01, \quad C = .1, \quad D = .001.
\]

Write down the corresponding system of differential equations.

4. Write down a system of difference equations

\[
x(t + 1) = x(t) + \cdots
\]
\[
y(t + 1) = y(t) + \cdots
\]

which could give numerical approximations to the solutions to the system (1) for the given choice of constants.

5. Use an Excel spreadsheet to find a numerical solution to the difference equation corresponding to the initial conditions

\[
x(0) = 100, \quad y(0) = 10.
\]

Work out your solution for 5 months into the future. To go further into the future, you should use Excel software to construct a spreadsheet. This should enable you to calculate the populations of rabbits and foxes for 30 months into the future. (This would be quite messy without the software.)

6. There is a constant solution to the differential system (1) with both fox and rabbit populations being positive. What can you say about the stability of this solution? What about the stability of the constant solutions to (2)?