Linear differential equations are differential equations that look like this:

\[ \frac{dy}{dt} + p_0(t)y = f(t), \]
\[ \frac{d^2y}{dt^2} + p_1(t)\frac{dy}{dt} + p_0(t)y = f(t), \]
\[ \frac{d^3y}{dt^3} + p_2(t)\frac{d^2y}{dt^2} + p_1(t)\frac{dy}{dt} + p_0(t)y = f(t), \]

and so forth. Of course, they can have even more derivatives. Note that the functions \( p_i(t) \) and \( f(t) \) are functions of only the single variable \( t \). These equations are said to be homogeneous if \( f(t) = 0 \).

Earlier, you have found some solutions to linear differential equations such as the equation

\[ \frac{dy}{dt} - ky = 0 \]

and \( \frac{d^2y}{dt^2} + y = 0 \)

by various ad hoc techniques. We would like to develop more systematic approaches to solving linear differential equations.

1. Show the following: If \( y_1(t) \) and \( y_2(t) \) are solutions to the homogeneous linear differential equation

\[ \frac{d^2y}{dt^2} + p_1(t)\frac{dy}{dt} + p_0(t)y = 0, \]  

then \( y_1(t) + y_2(t) \) is also a solution. If \( y_1(t) \) is a solution to (1) and \( c \) is a constant, then \( cy_1(t) \) is also a solution. If \( y_1(t) \) and \( y_2(t) \) are solutions to (1), then so is an arbitrary superposition \( c_1y_1(t) + c_2y_2(t) \). If \( y_1(t) \) is a solution to (1) and \( y_2(t) \) is a solution to

\[ \frac{d^2y}{dt^2} + p_1(t)\frac{dy}{dt} + p_0(t)y = f(t), \]  

then \( y_1(t) + y_2(t) \) is a solution to (2).
2. Check that \( y_1(t) = \cos t \) and \( y_2(t) = \sin t \) are solutions to the differential equation of simple harmonic motion

\[
\frac{d^2y}{dt^2} + y = 0.
\]

From this information, what is the most general possible solution to the differential equation of simple harmonic motion? Can you find the general solution to the differential equation

\[
\frac{d^2y}{dt^2} + y = 3e^t?
\]

Hint: First find a particular solution of the form \( y(t) = Ae^t \), and then use linearity. Can you find the general solution to

\[
\frac{d^2y}{dt^2} + y = 3t^2 + 2t - 5?
\]

3. Next, consider the differential equation

\[
\frac{dy}{dt} + p(t)y = f(t). \tag{3}
\]

We say that

\[
\frac{dy}{dt} + p(t)y = 0. \tag{4}
\]

is the associated first order differential equation. Show that the associated first order equation can always be solved by separation of variables. Find the general solution to the differential equation

\[
\frac{dy}{dt} + \frac{1}{t+1}y = 0,
\]

and write it in the form

\[
y(t) = cy_1(t), \quad \text{where } y_1(t) \text{ is a nonzero particular solution to (4)}.
\]

Here is a trick that can be used to solve (3) when we know a particular solution \( y_1(t) \) to (4). Set \( y(t) = v(t)y_1(t) \), substitute into (3) and solve for \( v(t) \). Use this idea to find the general solution to

\[
\frac{dy}{dt} + \frac{1}{t+1}y = \frac{t+2}{t+1}.
\]

4. Use the method described above to find the general solution to the differential equation

\[
\frac{dy}{dt} - \frac{1}{t+1}y = t^2, \quad t \geq 0.
\]
5. Can you find any solutions of the form \( y(t) = e^{\lambda t} \), where \( \lambda \) is constant, to the differential equation
\[
\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 0?
\]

What is the general solution to the differential equation
\[
\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 0?
\]

What is the general solution to the differential equation
\[
\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = e^{-t}?
\]

6. Can you find any solutions of the form \( y(t) = t^\lambda \), where \( \lambda \) is constant, to the differential equation
\[
t^2\frac{d^2y}{dt^2} - 2t\frac{dy}{dt} + 2y = 0?
\]

Note that you can divide by \( t^2 \) and rewrite this equation as
\[
\frac{d^2y}{dt^2} - \frac{2}{t}\frac{dy}{dt} + \frac{2}{t^2}y = 0,
\]
so that it is a linear homogeneous differential equation of the form (2) with 
\( p_1(t) = -2/t \) and \( p_2(t) = 2/t^2 \). What is the general solution to the differential equation
\[
t^2\frac{d^2y}{dt^2} - 2t\frac{dy}{dt} + 2y = 0?
\]