Nonlinear differential equations are often studied by **qualitative methods**. For example, consider the equation

\[ \frac{dy}{dt} = f(y) = y^2 - y^4. \]

We can also write this as

\[ \frac{dy}{dt} = -y^2(y - 1)(y + 1). \]

It has three constant solutions

\[ y = 0, \ y = -1, \ y = 1. \]

What happens as \( t \to \infty \)? It is helpful to imagine a fluid moving along the \( y \)-axis with stationary points at

\[ y = 0, \ y = -1, \ y = 1. \]
\[
\frac{dy}{dt} \quad \text{if } y > 1.
\]
\[
\frac{dy}{dt} \quad \text{if } 0 < y < 1.
\]
\[
\frac{dy}{dt} \quad \text{if } -1 < y < 0.
\]
\[
\frac{dy}{dt} \quad \text{if } y < -1.
\]
What are the stationary solutions for the ODE
\[
\frac{dy}{dt} = f(y) = y - y^2 - \frac{1}{4}.
\]
\[
y^2 - y + \frac{1}{4} = 0.
\]
SYSTEMS OF DIFFERENTIAL EQUATIONS:

The basic objects of study in the theory of differential equations are **first order autonomous systems** of differential equations:

\[
\begin{align*}
\frac{dx_1}{dt} &= f_1(x_1, x_2, \ldots, x_n), \\
\frac{dx_2}{dt} &= f_2(x_1, x_2, \ldots, x_n), \\
& \quad \ldots \\
& \quad \ldots \\
\frac{dx_n}{dt} &= f_n(x_1, x_2, \ldots, x_n).
\end{align*}
\]
A special case that illustrates many of the key concepts is a first order autonomous system of two equations:

\[ \frac{dx}{dt} = P(x, y), \]
\[ \frac{dy}{dt} = Q(x, y). \]

For example,

\[ \frac{dx}{dt} = x - xy, \]
\[ \frac{dy}{dt} = -y + xy. \]
A solution to the system
\[
\frac{dx}{dt} = P(x, y),
\]
\[
\frac{dy}{dt} = Q(x, y),
\]
is a pair of functions
\[
x = x(t), \quad y = y(t)
\]
such that
\[
\frac{d}{dt}(x(t)) = P(x(t), y(t)),
\]
\[
\frac{d}{dt}(y(t)) = Q(x(t), y(t)),
\]
For example,
\[ x(t) = \cos t, \quad y(t) = \sin t \]
is a solution to the system
\[ \frac{dx}{dt} = -y, \]
\[ \frac{dy}{dt} = x. \]

In fact
\[ x(t) = a \cos(t - t_0), \]
\[ y(t) = a \sin(t - t_0) \]
for any choice of constants \( a \) and \( t_0 \).
The second order differential equation
\[ \frac{d^2y}{dt^2} + y = 0 \] (1)
can be reduced to a first order system by reduction of order: Let
\[ x = \frac{dy}{dt}. \]
Then (1) is equivalent to the first order system
\[ \frac{dx}{dt} = -y, \]
\[ \frac{dy}{dt} = x. \]
All higher order systems can be reduced to first order systems by this method.
PICARD’S THEOREM. Suppose that the real-valued functions $P(x, y)$ and $Q(x, y)$ are well-behaved throughout the region $D$ in the $(x, y)$-plane. Then there is a unique maximal solution curve for the system of differential equations

\[
\frac{dx}{dt} = P(x, y),
\]
\[
\frac{dy}{dt} = Q(x, y).
\]

which passes through any point $(x_0, y_0)$ of $D$ at time $t = 0$.

(A “maximal” solution curve is one that cannot be extended without running out of $D$.)
Why is the theorem true?

Imagine a steady-state fluid flowing over the $(x, y)$-plane. Its velocity at the point $(x, y)$ is given by the vector

$$P(x, y)i + Q(x, y)j.$$ 

Now imagine the path $(x(t), y(t))$ of a particle in the fluid. The path will be a solution to the system

$$\frac{dx}{dt} = P(x, y),$$

$$\frac{dy}{dt} = Q(x, y).$$
Indeed, the velocity of the fluid particle is
\[ \mathbf{v}(t) = \frac{dx}{dt} \mathbf{i} + \frac{dy}{dt} \mathbf{j} \]
So the system of differential equations simply says that
\[ \mathbf{v}(t) = P(x(t), y(t)) \mathbf{i} + Q(x(t), y(t)) \mathbf{j}. \]
One can sometimes sketch the vector field
\[ P(x, y) \mathbf{i} + Q(x, y) \mathbf{j} \]
and get a qualitative picture of its solutions.
For example, we can find the explicit solutions to the system

\[ \frac{dx}{dt} = 2x, \]
\[ \frac{dy}{dt} = x + y. \]