SYSTEMS OF DIFFERENTIAL EQUATIONS:

The basic objects of study in the theory of differential equations are first order autonomous systems of differential equations. In the case of two unknowns, such a system takes the form

\[
\frac{dx}{dt} = P(x, y),
\]
\[
\frac{dy}{dt} = Q(x, y).
\]

where \( P \) and \( Q \) are functions of \( x \) and \( y \).

Picaard’s Theorem says that there is a unique solution passing through any given point, so long as \( P \) and \( Q \) are well-behaved.
Geometric interpretation: Imagine a steady-state fluid flowing over the $(x, y)$-plane whose velocity at the point $(x, y)$ is given by the vector $P(x, y)i + Q(x, y)j$.

If the path $t \mapsto (x(t), y(t))$ represents the motion of a particle in the fluid, then $(x(t), y(t))$ will be a solution to the system

$$
\frac{dx}{dt} = P(x, y), \\
\frac{dy}{dt} = Q(x, y).
$$
How many solutions are there to the system

\[
\left( \frac{dx}{dt} \right)^2 = x^2
\]

\[
\left( \frac{dy}{dt} \right)^2 = y^2
\]

which pass through the point \((1, 1)\) at time \(t = 0\)?
Sometimes we can find the explicit solutions to the system. For example,

\[
\frac{dx}{dt} = 2x,
\]

\[
\frac{dy}{dt} = x + y.
\]
But we frequently we need to use numerical methods together with a qualitative approach to determine properties of the solutions:

1. Determine the constant solutions or equilibria and determine whether they are stable.

2. Sketch the nullclines and determine regions in the \((x, y)\)-plane in which the fluid is flowing up or down, right or left.

3. Eliminate \(dt\) from the two equations and solve the resulting differential equation if possible.

4. The sketch of the solution curves is called the phase portrait of the system.
Find the constant solutions for the system

\[
\frac{dx}{dt} = y(y - 1),
\]

\[
\frac{dy}{dt} = x - 2.
\]
Find the constant solutions of the system

\[ \frac{dx}{dt} = -xy, \]

\[ \frac{dy}{dt} = xy - y. \]
Modeling bubonic plague:

Let $x(t)$ be the number of susceptible people at time $t$.

Let $y(t)$ be the number of sick people at time $t$.

Let $z(t)$ be the number of dead people at time $t$. 
We assume that the epidemic is invariably fatal and lasts for a short time, so that we can ignore births and deaths due to other causes.

It seems reasonable to assume that the number of new cases is proportional to \( xy \) and the number of deaths is proportional to \( y \). Then

\[
\frac{dx}{dt} = -Axy,
\]

\[
\frac{dy}{dt} = Axy - By.
\]

\[
\frac{dz}{dt} = By.
\]
We can ignore the last of the equation, because $z$ isn’t involved in the first two. We thus obtain the following system

\[
\frac{dx}{dt} = -Axy,
\]

\[
\frac{dy}{dt} = Axy - By,
\]

where $A > 0$ and $B > 0$.

What are the constant solutions?
What are the **nullclines**?

\[
\frac{dx}{dt} = 0 \quad \Rightarrow \\
\frac{dy}{dt} = 0 \quad \Rightarrow
\]
We can eliminate $t$ from this system to obtain
\[ \frac{dy}{dx} = -1 + \frac{B}{Ax}. \]
\[ y = -x + \frac{B}{A} \log |x| + c. \]
Note that when $x$ and $y$ are positive, $dx/dt < 0$. Thus the fluid flows to the left in the upper right quadrant.