Math 3C Lecture 15

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SYSTEMS OF DIFFERENTIAL EQUATIONS

Three methods for studying autonomous systems

\[
\frac{dx}{dt} = P(x, y),
\]

\[
\frac{dy}{dt} = Q(x, y).
\]

1. find explicit solutions when the system is sufficiently simple.
2. find numerical solutions.
3. determine the qualitative behavior of solutions
Simple systems of differential equations can be solved explicitly. For example, the system

\[
\frac{dx}{dt} = -x,
\]

\[
\frac{dy}{dt} = -2y.
\]

consists of two noninteracting equations. We can solve them separately to obtain the general solution

\[
x = c_1 e^{-t}, \quad y = c_2 e^{-2t}.
\]

note that there are two constants of integration in the general solution.
A given solution

\[ x = x(t), \quad y = y(t) \]

traces out a curve in the \((x, y)\)-plane, called a **trajectory** or an **orbit** for the solution curve. For example, since \( e^{-2t} = (e^{-t})^2 \), we can eliminate \( t \) from the previous solution to obtain

\[ y = c_2(e^{-t})^2 = c_2 \left( \frac{x}{c_1} \right)^2 = cx^2. \]

The trajectories for the system the system

\[
\frac{dx}{dt} = -x, \\
\frac{dy}{dt} = -2y.
\]

are just parabolas \( y = cx^2 \).
For example, consider the system
\[
\frac{dx}{dt} = -x, \\
\frac{dy}{dt} = -y.
\]
The constant solution at \((0, 0)\) is said to be \textit{asymptotically stable} in this case.
\[
\frac{dx}{dt} = x, \\
\frac{dy}{dt} = y.
\]

In this case, the constant solution at \((0, 0)\) is said to be **unstable**.
The phase portrait for a nonlinear system can often be obtained by the procedure:

1. Determine the **constant solutions** or **equilibria** and determine whether they are **stable** if possible.

2. Sketch the **nullclines** and determine regions in the $(x, y)$-plane in which the fluid is flowing up or down, right or left.

3. Eliminate $dt$ from the two equations and solve the resulting differential equation if possible.

4. Then sketch the solution curves.
Imagine an island inhabited by foxes and rabbits, one species (the foxes) depending upon the other (the rabbits) for food.

Let

$x(t) = \text{number of rabbits at time } t,$

$y(t) = \text{number of foxes at time } t,$

There is unlimited food for the rabbits so in the absence of foxes they grow by the exponential law:

$$\frac{dx}{dt} = ax, \quad a > 0.$$
But when foxes are present, the number of rabbits eaten by foxes is proportional to the number of rabbits, but also proportional to the number of foxes:

\[
\frac{dx}{dt} = ax - bxy, \quad b > 0.
\]

With no rabbits present, the foxes would have to scrounge for other types of food, and their death rate would exceed their death rate:

\[
\frac{dy}{dt} = -cy, \quad c > 0.
\]

But if rabbits are present, the birth rate of foxes goes up at a rate proportional to the number of rabbits:

\[
\frac{dy}{dt} = -cy + dxy, \quad d > 0.
\]
The system of differential equations
\[
\frac{dx}{dt} = ax - bxy,
\]
\[
\frac{dy}{dt} = -cy + dxy,
\]
is known as the Volterra-Lotka predator-prey equations. If we set \( a = b = c = d = 1 \), they become
\[
\frac{dx}{dt} = x - xy,
\]
\[
\frac{dy}{dt} = -y + xy,
\]
What are the equilibria or constant solutions?
The **horizontal nullclines** are the points at which \( \frac{dy}{dt} = 0 \):

The **vertical nullclines** are the points at which \( \frac{dx}{dt} = 0 \):
Given the constant solutions \((0, 0)\) and \((1, 1)\) and the information on nullclines it is possible to give a very rough sketch of the phase portrait:
Can we solve for orbits in the $(x, y)$-plane?

\[
\frac{dy}{dx} = \frac{-y + xy}{x - xy} = \frac{(-1 + x)y}{x(1 - y)}.
\]

\[
\frac{1-y}{y} dy = - \frac{1-x}{x} dx,
\quad \left( \frac{1}{y} - 1 \right) dy = - \left( \frac{1}{x} - 1 \right) dx
\]

\[
\log |y| - y = -(\log |x| - x) + c,
\log |x| - y + \log |y| - y = c.
\]

Let \( f(u) = \log |u| - u \). Then the equation for the trajectories is

\[
F(x, y) = f(x) + f(y) = c.
\]
What are the maxima and minima of \( f(u) \)?

\[
f'(u) = \frac{1}{u} - 1 = 0 \quad \Rightarrow \quad u = 1.
\]

\[
f''(u) = -\frac{1}{u^2} < 0.
\]

Therefore \( u = 1 \) is a local maximum for \( f(u) \), \( x = 1 \) is a local maximum for \( f(x) \), and \( y = 1 \) is a local maximum for \( f(y) \).

It follows that \( (1, 1) \) is a local maximum for

\[
F(x, y) = f(x) + f(y).
\]

The orbits of the predator-prey equations are \textbf{level sets} for the topographic map of \( F(x, y) \). On this topographic map \( (1, 1) \) is a local maximum, so \( (1, 1) \) represents a mountain peak. Nearby orbits are closed and \( (1, 1) \) is \textbf{conditionally stable}. 