TENTATIVE FORMULA I
Midterm I: 20%
Midterm II: 20%
Homework: 10%
Quizzes: 10%
Final: 40%

TENTATIVE FORMULA II
Higher of two midterms: 30%
Homework: 10%
Quizzes: 10%
Final: 50%
INVERSES AND DETERMINANTS

Matrices represent linear transformations. The function

\[
\begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix} \mapsto \begin{pmatrix}
  y_1 \\
  y_2
\end{pmatrix} = \begin{pmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
\end{pmatrix} \begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix}
\]

\[x \mapsto y = Ax\]

is called a linear transformation.
The $n \times n$ matrix $B$ is said to be the inverse of an $n \times n$ matrix $A$ (and $A$ is the inverse of $B$) if

$$BA = I \quad \text{or} \quad AB = I,$$

where $I$ is the $n \times n$ identity matrix

$$I = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix}.$$ 

We also say that the linear transformation

$$\mathbf{y} \mapsto \mathbf{x} = B\mathbf{y}$$

is the inverse of the linear transformation

$$\mathbf{x} \mapsto \mathbf{y} = A\mathbf{x}.$$
A matrix is invertible if it has an inverse. How does one find the inverse to an invertible matrix?

The idea is simple: We try to write the equation

\[ Ax = y \]

in the form

\[ x = By \]

by means of elementary operations on equations.
Elementary operations on equations:

- Interchange two equations.
- Multiply an equation by a nonzero constant.
- Add a constant multiple of one equation to another.

Elementary row operations on matrices:

- Interchange two rows of the matrix.
- Multiply a row by a nonzero constant.
- Add a constant multiple of one row to another.
We start with the linear system
\[ a_{11}x_1 + a_{12}x_2 = y_1, \]
\[ a_{21}x_1 + a_{22}x_2 = y_2, \]
and use the elementary operations on equations to put it in the form
\[ x_1 = b_{11}y_1 + b_{12}y_2, \]
\[ x_2 = b_{21}y_1 + b_{22}y_2. \]

Equivalently, we start with the matrix
\[
\begin{pmatrix}
  a_{11} + a_{12} & 1 & 0 \\
  a_{21} + a_{22} & 0 & 1
\end{pmatrix}
\]
and use the elementary operations on matrices to put it in the form
\[
\begin{pmatrix}
  1 & 0 & b_{11} & b_{12} \\
  0 & 1 & b_{21} & b_{22}
\end{pmatrix}.
\]
For example, let’s find the inverse of the matrix

\[ A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}. \]

\[ (\begin{array}{cc} 2 \\ -1 \end{array}, \begin{array}{c} -1 \\ 1 \end{array}) \]
Suppose we want to solve the linear system

\[ x + y = 4, \]
\[ x + 2y = 3. \]

We can write this as

\[ A \mathbf{x} = \mathbf{b}, \quad \text{where} \]
\[ A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}. \]

We can multiply on the left by \( A^{-1} \), the inverse of \( A \), to obtain
Recall that one can use the elementary row operations to put any matrix in **row-reduced echelon form**:

- All zero rows are at the bottom of the matrix.
- The first nonzero entry in any nonzero row, called its **pivot**, is a one.
- The pivots in lower rows are to the right of pivots for higher rows.
- If a column contains an initial one for some row, all the other entries in the column are zero.
Procedure for inverting an $n \times n$ matrix $A$

1. Form the $n \times (2n)$ matrix $(A|I)$ where $I$ is the identity matrix.
2. Put the matrix in row-reduced echelon form.
3. If the first $n$ elements in any row are zero, the matrix is NOT invertible.
4. Otherwise, the row-reduced echelon form is $(I|B)$, where $B$ is the inverse of $A$. 
The linear transformation
\[
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\]
takes a plane to a straight line. It does not have an inverse.
Associated to any $n \times n$ matrix is a number called its determinant and denoted by $|A|$.

For $2 \times 2$ matrices there is a simple formula for the determinant:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

For example,

$$\begin{vmatrix} 3 & 2 \\ 5 & 4 \end{vmatrix} =$$
The absolute value of the determinant
\[
\begin{vmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
\end{vmatrix}
\]
is the area of the parallelogram spanned by its rows

Indeed
\[
\begin{vmatrix}
  a \cos \theta & a \sin \theta \\
  b \cos \phi & b \sin \phi
\end{vmatrix}
= ab(\cos \theta \sin \phi - \sin \theta \cos \phi)
= ab \sin(\phi - \theta).
\]
There is also a formula for determinants of $3 \times 3$ matrices:

\[
\begin{vmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{33} \\
  a_{31} & a_{32} & a_{33}
\end{vmatrix}
\]

\[
= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}.
\]

Here we have expanded in terms of the first column. Formulae for determinants of $n \times n$ determinants are given in the text, but it may be easier to simply apply the axioms to evaluate determinants.
AXIOMS FOR DETERMINANTS

1. If $B$ is obtained from $A$ by interchanging two rows, $|B| = -|A|$.
2. If $B$ is obtained from $A$ by adding a constant multiple of one row to another, $|B| = |A|$.
3. If a row of $A$ is multiplied by a nonzero scalar $k$ to obtain $B$, then $|B| = k|A|$.
4. If $A$ is the identity matrix, then $|A| = 1$.
5. $A$ contains a row of zeros, then $|A| = 0$.  

The axioms allow us to calculate the determinant by means of the elementary row operations. For large matrices, this is usually easier than expanding in terms of a row or column.

\[
\begin{vmatrix}
1 & 2 & 2 & 3 \\
2 & 9 & 4 & 6 \\
1 & 2 & 2 & 4 \\
1 & 2 & 5 & 3 \\
\end{vmatrix} = 
\]