A vector $\mathbf{v}$ is a **linear combination** of the vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ if there are scalars $c_1, \ldots, c_k$ such that

$$\mathbf{v} = c_1 \mathbf{v}_1 + \cdots + c_k \mathbf{v}_k.$$ 

A collection of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ in a vector space $V$ is **linearly dependent** if one of the vectors can be written as a linear combination of the others. The collection $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is **linearly independent** if not linearly dependent.
Suppose

\[ \mathbf{v}_1 = \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} 8 \\ 4 \\ 6 \end{pmatrix}, \]

both elements lying in \( \mathbb{R}^3 \). Is the collection \( \{ \mathbf{v}_1, \mathbf{v}_2 \} \) linearly dependent or independent?

\[
\begin{pmatrix} 8 \\ 4 \\ 6 \end{pmatrix} = 2 \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix},
\]

so these vectors are linearly dependent.
Suppose
\[ \mathbf{v}_1 = \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}. \]

Is the collection \( \{ \mathbf{v}_1, \mathbf{v}_2 \} \) linearly dependent or independent?

If they are linearly dependent, 
\[ \mathbf{v}_1 = a\mathbf{v}_2 \quad \text{or} \quad \mathbf{v}_2 = b\mathbf{v}_1. \]
in either case there is a nontrivial solution \( (c_1, c_2) \) to the equation
\[ c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}. \]

\[ 4c_1 + c_2 = 0, \]
\[ 2c_1 = 0, \]
\[ 3c_1 + 5c_2 = 0. \]

Only solution is \( c_1 = c_2 = 0 \), so the vectors are linearly independent.
It is helpful to have criteria for determining when a collection \( \{v_1, \ldots, v_k\} \) of vectors in a vector space \( V \) is linearly dependent or independent.

One such criterion is: \( \{v_1, \ldots, v_k\} \) are linearly dependent if and only if there is a nontrivial solution \((c_1, \ldots, c_k)\) to the equation

\[
c_1v_1 + \cdots + c_kv_k = 0.
\]

Are the vectors

\[
v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}
\]

linearly dependent or independent?
Does the system
\[\begin{align*}
c_1 + 3c_2 + c_3 &= 0, \\
c_2 + c_3 &= 0, \\
3c_3 &= 0
\end{align*}\]
have a nontrivial solution? The last equation implies \(c_3 = 0\), then the second equation implies \(c_2 = 0\) and finally the first equation implies that \(c_1 = 0\). Therefore the three vectors
\[
v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}
\]
are linearly independent.
Are the vectors 
\[ \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 4 \\ 5 \\ 9 \end{pmatrix} \]
linearly dependent or independent?

Does the system 
\[
\begin{align*}
    c_1 + 3c_2 + 4c_3 &= 0, \\
    c_1 + 4c_2 + 5c_3 &= 0, \\
    2c_1 + 7c_2 + 9c_3 &= 0
\end{align*}
\]
have a nontrivial solution? We can use the elementary row operations to find out.
The **span** of a collection of vectors \( \{v_1, \ldots, v_k\} \) in a vector space \( V \) is the set

\[
\text{Span}\{v_1, \ldots, v_k\}
\]

of all linear combinations of the vectors. The set

\[
W = \text{Span}\{v_1, \ldots, v_k\}
\]

is a subspace of \( V \).
Let $V$ be a vector space.

A **basis** for $V$ is a collection of vectors $\{v_1, \ldots, v_k\}$ such that

1. $V = \text{Span}\{v_1, \ldots, v_k\}$, and
2. $\{v_1, \ldots, v_k\}$ are linearly independent.

In simple terms, this means that any element of $x \in V$ can be written as

$$x = c_1v_1 + \cdots + c_kv_k.$$ 

and the representation is unique. We can think of $c_1, \ldots, c_k$ as the **coordinates** of $x$ with respect to the basis.
The **dimension** of a vector space $V$ is the number of elements in any basis. (It is a theorem that the number of elements in any basis is always the same.)

The standard basis for $\mathbb{R}^n$ is

$$
e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

Since this is a basis, the dimension of $\mathbb{R}^n$ is $n$.

Any vector $x \in \mathbb{R}^n$ can be written uniquely as a linear combination of $e_1, \ldots, e_n$:

$$
\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + x_n \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}
= x_1 e_1 + \cdots + x_n e_n.
$$
Suppose that
\[ \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}. \]

Is \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \} a basis for \( \mathbb{R}^3 \)?

No, this collection is not linearly independent because
\[ 2\mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}. \]

Every basis for \( \mathbb{R}^3 \) must be linearly independent and must span \( \mathbb{R}^3 \).
**Theorem.** A collection of $n$ vectors

$$
\mathbf{a}_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{pmatrix}, \\
\cdots, \quad \mathbf{a}_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{pmatrix}
$$

is a basis for $\mathbb{R}^n$ if and only if the matrix

$$
A = \begin{pmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n1} & a_{n2} & \cdots & a_{nn} 
\end{pmatrix}
$$

is invertible.
Suppose that
\[ \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}. \]

Is \( \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \) a basis for \( \mathbb{R}^3 \)?
Suppose that
\[ W = \text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\} \subseteq \mathbb{R}^5 \]
where
\[ \mathbf{a}_1 = (1, 1, 0, 2, 3), \]
\[ \mathbf{a}_2 = (0, 0, 1, 1, 2), \]
\[ \mathbf{a}_3 = (1, 1, 1, 3, 5). \]
Can we find a basis for \( W \)? What is the dimension of \( W \)?

\( W \) is the **row space** of the matrix
\[
A = \begin{pmatrix}
1 & 1 & 0 & 2 & 3 \\
0 & 0 & 1 & 1 & 2 \\
1 & 1 & 1 & 3 & 5
\end{pmatrix}.
\]
Elementary row operations do not change the row space of the matrix.

**Method for finding a basis for the row space $W$ of a matrix:**

Perform elementary row operations to put the matrix in row-reduced echelon form. Then the **nonzero rows** of the row-reduced echelon form make up a basis for $W$. The number of elements in the resulting basis is the dimension of $W$. 
One of the major applications of linear algebra consists of understanding the space of solutions to a homogeneous linear system

\[
\begin{align*}
  a_{11}x_1 &+ a_{12}x_2 + \cdots + a_{1n}x_n = 0, \\
  a_{21}x_1 &+ a_{22}x_2 + \cdots + a_{2n}x_n = 0, \\
          &\vdots \quad \vdots \quad \cdots \quad \cdots \quad \vdots \\
  a_{m1}x_1 &+ a_{m2}x_2 + \cdots + a_{mn}x_n = 0
\end{align*}
\]

If we set

\[
\begin{align*}
  \mathbf{a}_1 &= (a_{11}, a_{12}, \cdots, a_{1n}), \\
  \mathbf{a}_2 &= (a_{21}, a_{22}, \cdots, a_{2n}), \\
          &\quad \vdots \\
  \mathbf{a}_m &= (a_{m1}, a_{m2}, \cdots, a_{mn}),
\end{align*}
\]

we can rewrite this system as

\[
\begin{align*}
  \mathbf{a}_1 \cdot \mathbf{x} &= 0, \\
  \mathbf{a}_2 \cdot \mathbf{x} &= 0, \\
          &\quad \vdots \\
  \mathbf{a}_m \cdot \mathbf{x} &= 0.
\end{align*}
\]
Let \( W = \text{Span}\{ \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m \} \). Then

the space of solutions to

\[
\begin{align*}
 a_{11}x_1 + a_{12}x_2 & \cdots + a_{1n}x_n = 0, \\
 a_{21}x_1 + a_{22}x_2 & \cdots + a_{2n}x_n = 0, \\
 \quad \vdots \quad & \vdots \quad \quad \quad \vdots \\
 a_{m1}x_1 + a_{m2}x_2 & \cdots + a_{mn}x_n = 0
\end{align*}
\]

is the collection of vectors which are perpendicular to

\( \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m \).

We denote this space by \( W^\perp \) and call it the **orthogonal complement** to \( W \). We can find bases for \( W \) and \( W^\perp \) by using the elementary row operations on the matrix

\[
egin{pmatrix}
 a_{11} & a_{12} & \cdots & a_{1n} \\
 a_{21} & a_{22} & \cdots & a_{2n} \\
 \vdots & \vdots & \ddots & \vdots \\
 a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\]