Math 3C Lecture 6

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PICARD’S THEOREM. Suppose that the real-valued function $f(t, y)$ is well-behaved throughout the region $D$ in the $(t, y)$-plane. Then there is a unique maximal solution curve for the differential equation

$$\frac{d}{dt}(y(t)) = f(t, y(t))$$

which passes through every point $(t_0, y_0)$ of $D$.

(A “maximal” solution curve is one that cannot be extended without running out of $D$.

A differential equation is in \textbf{standard form} if it is written in the form

$$\frac{dy}{dt} = f(t, y)$$
consider
\[
\left( \frac{dy}{dt} \right)^2 = y^2.
\]
To apply Picard’s Theorem we need to \textbf{solve for} \(dy/dt\) and put the differential equation in \textbf{canonical form}. But when we take the square root, the differential equation breaks into two,\[
\frac{dy}{dt} = y, \quad \frac{dy}{dt} = -y.
\]
The solutions to the first of these equations are \(y = ce^t\), while the solutions to the second of the equations are \(y = ce^{-t}\). Thus through most points in the \((t, y)\)-plane, there are two solutions, one exponentially growing, the other exponentially decaying.

Most differential equations cannot be solved exactly by integration. In these cases one must use numerical methods to find approximate solutions.
Today we will consider the Euler polygon method for finding approximate solutions to differential equations.

We seek a solution which goes through the point \((t_0, y_0)\). The solution will be approximated by a straight line of slope \(f(t_0, y_0)\). Let \(h\) be a small number.

Then

\[
(t_1, y_1) = (t_0 + h, y_0 + hf(t_0, y_0))
\]

will lie very close to the exact solution which passes through the point \((t_0, y_0)\).
We can repeat the process starting at the point \((t_1, y_1)\). We move along a small line segment starting at \((t_1, y_1)\) of slope \(f(t_1, y_1)\) to a point 
\[
(t_2, y_2) = (t_1 + h, y_1 + hf(t_1, y_1)).
\]

We then do the same thing at \((t_2, y_2)\) and so forth. At the \(i\)-th stage we move from \((t_i, y_i)\) to 
\[
(t_{i+1}, y_{i+1}) = (t_i + h, y_i + hf(t_i, y_i)).
\]
We thus obtain a polygon from \((t_0, y_0)\) to \((t_1, y_1)\) to \((t_2, y_2)\) and so forth to \((t_n, y_n)\), which approximates the solution to the differential equation. This is called the **Euler polygon** of step size \(h\) to the solution.
Problem: Find an approximation to $y(1)$ if $y(t)$ is the solution to

$$\frac{dy}{dt} = 2ty, \quad y(0) = 1.$$  

We set $h = 0.5$. In this case $(t_0, y_0) = (0, 1)$, and we can find the other points of the polygon by the formulae

$$t_{i+1} = t_i + h,$$

$$y_{i+1} = y_i + hf(t_i, y_i) = y_i + h2t_iy_i.$$

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\[
\frac{dy}{dt} = 2ty, \quad y(0) = 1.
\]
Suppose instead we set \( h = 0.2 \).

\[
t_{i+1} = t_i + h,
\]

\[
y_{i+1} = y_i + hf(t_i, y_i) = y_i + h2t_iy_i.
\]

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If one carries out the calculations one obtains an approximation for \( y(1) \): 2.05058
When applying a numerical method there are two types of error:

1. *discretization error* which results from the step size $h$ being too large.

2. *roundoff error* which results from the fact that calculators and computers have only a certain number of digits of precision.

As we choose smaller and smaller values of $h$ the discretization error gets smaller, but the roundoff error increases.