Suppose that the real-valued function \( f(t, y) \) is well-behaved. Then Picard’s Theorem says that there is a unique solution to the initial value problem

\[
\frac{d}{dt} (y(t)) = f(t, y(t)), \quad y(t_0) = y_0.
\]

One can find approximate solutions to this initial value problem by means of the Euler polygon method. To apply this method, we choose a small number \( h \). Start at

\[(t_0, y_0)\]

and let

\[(t_1, y_1) = (t_0 + h, y_0 + hf(t_0, y_0)), \]
\[(t_2, y_2) = (t_1 + h, y_1 + hf(t_1, y_1)), \]

and so forth.
At the $i$-th stage we move from $(t_i, y_i)$ to

$$(t_{i+1}, y_{i+1}) = (t_i + h, y_i + hf(t_i, y_i)).$$

Thus we obtain a polygon from $(t_0, y_0)$ to $(t_1, y_1)$ to $(t_2, y_2)$ to $\cdots$ to $(t_n, y_n)$, called the **Euler polygon** approximation of step size $h$.

From the polygon, one obtains a table of values for an approximation to the solution to the initial value problem.
Problem: Find an approximation to $y(1)$ if $y(t)$ is the solution to

$$\frac{dy}{dt} = t + y, \quad y(0) = 0.$$  

We set $h = 0.5$. In this case $(t_0, y_0) = (0, 0)$, and we can find the other points of the polygon by the formulae

$$t_{i+1} = t_i + h = t_i + .5,$$

$$y_{i+1} = y_i + hf(t_i, y_i) = y_i + .5(t_i + y_i).$$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$t_i$</th>
<th>$y_i$</th>
<th>$f(t_i, y_i)$</th>
<th>$hf(t_i, y_i)$</th>
</tr>
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<td>2</td>
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Suppose instead we set $h = 0.2$.

$$t_{i+1} = t_i + 0.2,$$

$$y_{i+1} = y_i + hf(t_i, y_i) = y_i + 0.2(t_i + y_i).$$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$t_i$</th>
<th>$y_i$</th>
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<td>5</td>
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</table>

As $h$ gets smaller and smaller one gets better approximations to $y(1)$.

The exact solution to the initial-value problem is $y(t) = e^t - t - 1$, so $y(1) = e - 2$. 
If \( y(t) \) is the exact solution to the differential equation, Taylor’s theorem from calculus states that

\[
y(t + h) = y(t) + hy'(t) + \frac{h^2}{2}y''(t^*),
\]

where \( t^* \) is some number within the interval \((t, t + h)\). Since \( y(t) \) is a solution to the differential equation

\[
y'(t) = f(t, y(t)),
\]

we see that

\[
y(t + h) = y(t) + hf(t, y) + \frac{h^2}{2}y''(t^*).
\]

The **local discretization error** for Euler’s method is therefore

\[
\frac{h^2}{2}y''(t^*).
\]
To get from $y(0)$ to $y(1)$ we must take $1/h$ steps, so we say that the global discretization error is of order $h^2/h$, or of order $h$. We say that the error is of order one and that Euler’s method is $O(h)$.

Other methods have smaller discretization error, such as the modified Euler method:

$$t_{i+1} = t_i + h,$$

$$y_{i+1} = y_i + \frac{h}{2} \left[ f(t_i, y_i) + f(t_{i+1}, y_i + h f(t_i, y_i)) \right].$$

This method is $O(h^2)$. 
We can apply Euler’s method to an initial value problem for the equation of exponential growth or decay:

\[ \frac{dy}{dt} = ky, \quad y(0) = 1. \]

What is \( y(1) \)?

Here the equations we must solve are:

\[ t_{i+1} = t_i + h, \]
\[ y_{i+1} = y_i + hf(t_i, y_i) = y_i + hky_i. \]
Suppose we divide the interval \([0, 1]\) into \(n\) equal pieces, and let \(h = 1/n\), where \(n\) is a positive integer. Then the second of these equations becomes

\[
y_{i+1} = \left(1 + \frac{k}{n}\right)y_i.
\]

We start with \(y_0 = 1\). Then

\[
y_1 = \left(1 + \frac{k}{n}\right).
\]

\[
y_n = \left(1 + \frac{k}{n}\right)^n.
\]
Thus we see that the Euler approximation to $e = y(1)$ is

$$y_n = \left(1 + \frac{k}{n}\right)^n,$$

an approximation which gets better and better as $n \to \infty$. Since Euler’s method converges to the exact value $e^k$ as $h = 1/n \to 0$, we recover an important limit

$$e^k = \lim_{n \to \infty} \left(1 + \frac{k}{n}\right)^n.$$
The Euler method for solving the equation
\[
\frac{dy}{dt} = ky
\]
can be interpreted as compounding of interest. We imagine that \(y(t)\) is the amount of indebtedness of a credit card account at time \(t\), the interest rate being \(k\). We suppose that the bank would like to maximize its rate of return by compounding sufficiently frequently that the amount of indebtedness at the end of one year will be a good approximation to \(e^k\). How often should the bank compound?

Quarterly? Monthly? Daily?
Let us suppose that the interest rate on the credit card is 18\%, so \( k = .18 \). With no compounding, $100 of debt will grow to \((1 + .18)(100)\) or $118 in one year.

Quarterly compounding corresponds to \( n = 4 \). In this case,
\[
\left(1 + \frac{.18}{4}\right)^4 = 1.19252,
\]
so a debt of $100 grows to a debt of $119.25.

Monthly compounding corresponds to \( n = 12 \). In this case,
\[
\left(1 + \frac{.18}{12}\right)^{12} = 1.19562,
\]
so a debt of $100 grows to a debt of $119.56.
Daily compounding corresponds to $n = 365$. In this case,

$$\left(1 + \frac{.18}{365}\right)^{365} = 1.19716,$$

so a debt of $100 grows to a debt of $119.72.

Since

$$e^{(.18)} = 1.19722,$$

under instantaneous compounding a debt of $100 grows to a debt of $119.72.

The Euler method gives a discrete version of exponential growth which has an important application—compounding of interest.