It was gradually found that the easiest way to present theory of limits needed for the foundation of calculus uses the notion of open subset of the space $\mathbb{R}$ of real numbers. The family of such open subsets is called the standard topology for the real numbers. In full generality, a topology on a set $X$ is a collection $T$ of subsets of $X$ such that

1. the empty set $\emptyset$ and the whole space $X$ are elements of $T$,
2. the union of an ARBITRARY collection of elements of $T$ is a member of $T$, and
3. the intersection of a FINITE collection of elements of $T$ is a member of $T$.

This is a very abstract definition, so for the time being the reader may want to focus instead on the notion of open sets within $\mathbb{R}$, which will be developed in the following pages. The family $T$ of open sets in $\mathbb{R}$ is the simplest example of topology. Gradually, it will become apparent that this topology is useful for formulating the various notions of limit, and we will see how these limits are used to define continuous functions. Ultimately, they are also used to define derivatives of functions.

In this installment of the lecture notes, we highlight the most important topics presented in Chapter 3 of the text [1], and provide a few additional topics on metric spaces, in the hopes of providing an easier transition to more advanced books on real analysis, such as [3]. The study of the various topologies that one can define on sets has developed into an important branch of mathematics in its own right, called topology [2].

1 Open and closed sets

First, some commonly used notation. If $a$ and $b$ are real numbers with $a < b$, we let

$$(a, b) = \{ x \in \mathbb{R} : a < x < b \}, \quad [a, b] = \{ x \in \mathbb{R} : a \leq x \leq b \}.$$  

Moreover, if $x \in \mathbb{R}$ and $\epsilon > 0$, we let

$$N(x; \epsilon) = \{ y \in \mathbb{R} : |x - y| < \epsilon \}.$$
and call this the open $\epsilon$-ball about $x$, and let

$$N^*(x;\epsilon) = \{y \in \mathbb{R} : 0 < |x - y| < \epsilon\},$$

which we call the deleted open $\epsilon$-ball about $x$. Notice that

$$N(x;\epsilon) = (x - \epsilon, x + \epsilon).$$

**Definition.** A set $S \subseteq \mathbb{R}$ is open if whenever $x \in S$, there exists a real number $\epsilon > 0$ such that $N(x;\epsilon) \subseteq S$.

Examples of open sets include $(a,b)$ when $a < b$, as well as the open $\epsilon$-ball $N(x;\epsilon)$ about $x$ and the deleted open $\epsilon$-ball $N^*(x;\epsilon)$ about $x$. If we let

$$T = \{U \in \mathcal{P}(\mathbb{R}) : U \text{ is open }\},$$

then the following proposition states that $T$ is a topology on $\mathbb{R}$:

**Proposition 1.1.** The empty set $\emptyset$ and the whole space $\mathbb{R}$ are open. The union of an arbitrary collection of open sets is open. The intersection of a finite collection of open sets is open.

Proof: We leave it to the reader to check that the empty set $\emptyset$ and the whole space $\mathbb{R}$ are open. Suppose that $\{U_\alpha : \alpha \in A\}$ is a collection of open sets and that

$$U = \bigcup\{U_\alpha : \alpha \in A\}.$$

If $x \in U$, then $x \in U_\alpha$ for some $\alpha \in A$. Hence there is an $\epsilon > 0$ such that $N(x;\epsilon) \subseteq U_\alpha$. But then $N(x;\epsilon) \subseteq U$, and this shows that $U$ is open.

Suppose that $\{U_1, \ldots, U_m\}$ is a finite collection of open sets and that

$$U = U_1 \cap \cdots \cap U_m.$$

If $x \in U$, then $x \in U_i$ for every $i$, $1 \leq i \leq m$. Hence for each $i$, $1 \leq i \leq m$, there exists $\epsilon_i > 0$ such that $N(x;\epsilon_i) \subseteq U_i$. Let

$$\epsilon = \min(\epsilon_1, \ldots, \epsilon_m),$$

and note that $\epsilon > 0$ since the minimum of a finite number of positive numbers is positive. Then $N(x;\epsilon) \subseteq U_i$ for every $i$, $1 \leq i \leq m$. Hence $N(x;\epsilon) \subseteq U$ and $U$ is open.

**Definition.** A set $S \subseteq \mathbb{R}$ is closed if $\mathbb{R} - S$ is open.

Examples of closed sets include the closed intervals $[a,b]$ when $a < b$. Any finite subset of $\mathbb{R}$ is also closed. Many of the properties of closed sets can be derived from the corresponding properties of open sets, as illustrated by the proof of:

**Proposition 1.2.** The empty set $\emptyset$ and the whole space $\mathbb{R}$ are closed. The intersection of an arbitrary collection of closed sets is closed. The union of a finite collection of closed sets is closed.
Proof: Once again it is easy to see that the empty set $\emptyset$ and the whole space $\mathbb{R}$ are closed. Suppose that $\{C_\alpha : \alpha \in A\}$ is a collection of closed sets and that

$$C = \bigcap \{C_\alpha : \alpha \in A\}.$$

Then $U_\alpha = \mathbb{R} - C_\alpha$ is open, and by the previous proposition

$$U = \bigcup \{U_\alpha : \alpha \in A\}$$

is open. It follows from the De Morgan laws that

$$\mathbb{R} - U = \mathbb{R} - \bigcup \{U_\alpha : \alpha \in A\} = \bigcap (\mathbb{R} - U_\alpha) = \bigcap \{C_\alpha : \alpha \in A\} = C.$$

Hence $C$ is closed.

Suppose that $\{C_1, \ldots, C_m\}$ is a finite collection of open sets and that

$$C = C_1 \cup \cdots \cup C_m.$$

Then $U_i = \mathbb{R} - C_i$ is open and by the previous proposition, $U = U_1 \cap \cdots \cap U_m$ is also open. It follows from the De Morgan laws that

$$\mathbb{R} - U = \mathbb{R} - (U_1 \cap \cdots \cap U_m)$$

$$= (\mathbb{R} - U_1) \cap \cdots \cap (\mathbb{R} - U_m) = C_1 \cup \cdots \cup C_m = C,$$

and hence $C$ is closed.

Although open and closed sets are the primary concepts, there is some closely related terminology which is often quite useful.

**Definition.** Suppose $S \subset \mathbb{R}$. Then the interior of $S$ is

$$\text{int}(S) = \{x \in S : N(x; \epsilon) \subseteq S \text{ for some } \epsilon > 0 \}.$$ 

The boundary of $S$ is

$$\text{bd}(S) = \mathbb{R} - (\text{int}(S) \cup \text{int}(\mathbb{R} - S)).$$

Finally, the closure of $S$ is

$$\text{Cl}(S) = S \cup \text{bd}(S).$$

You can easily check that $\mathbb{R}$ is the disjoint union of $\text{int}(S)$, $\text{int}(\mathbb{R} - S)$ and $\text{bd}(S)$. Clearly, $S$ is open if and only if $S = \text{int}(S)$, and the following proposition shows that $S$ is closed if and only if $S = \text{Cl}(S)$:

**Proposition 1.3.** $S$ is closed if and only if $\text{bd}(S) \subseteq S$.

Proof: $\Rightarrow$: Suppose that $S$ is closed. Then $\mathbb{R} - S$ is open so $\text{int}(\mathbb{R} - S) = \mathbb{R} - S$. Thus $S = \text{int}(S) \cup \text{bd}(S)$, and $\text{bd}(S) \subseteq S$. 

3
Proof: Suppose that \( \text{bd}(S) \subseteq S \). Then \( \text{int}(S) \cup \text{bd}(S) \subseteq S \), so \( \mathbb{R} - S \subseteq \overline{f(\mathbb{R} - S)} \).

Hence \( \text{int}(\mathbb{R} - S) = \mathbb{R} - S \) and \( S \) is closed.

The preceding propositions allow one to check whether many subsets of \( \mathbb{R} \) are open or closed. For example, consider the subset \( \mathbb{Z} \subseteq \mathbb{R} \). Since \( \mathbb{R} - \mathbb{Z} \) is the union of the intervals \((n, n+1)\) for \( n \in \mathbb{Z} \), we see that \( \mathbb{R} - \mathbb{Z} \) is open, and hence \( \mathbb{Z} \) is closed.

On the other hand we claim that the rationals \( \mathbb{Q} \subseteq \mathbb{R} \) is neither open nor closed. Equivalently, neither \( \mathbb{Q} \) nor \( \mathbb{R} - \mathbb{Q} \) is open.

If \( \mathbb{Q} \) were open there would exist \( \epsilon > 0 \) such that \((-\epsilon, \epsilon) \subseteq \mathbb{Q}\). But then there would exist \( n \in \mathbb{N} \) such that \((-1/n, 1/n) \subseteq \mathbb{Q}\). Since multiplication by \( n \) takes \( \mathbb{Q} \) to itself, it would then follow that \((-1, 1) \subseteq \mathbb{Q}\). This is impossible, because we have seen that \((-1, 1)\) is uncountable while \( \mathbb{Q} \) is countable.

To show that \( \mathbb{R} - \mathbb{Q} \) is not open, we use:

**Proposition 1.4.** Any nonempty open interval \((a, b)\) contains a rational number \( r \).

Proof: In proving this, we can assume without loss of generality (WOLOG) that \( 2 < a < b \). (Indeed, we could arrange this by replacing \((a, b)\) by \((a + n, b + n)\) for some \( n \in \mathbb{N} \).) By the Archimedean property of \( \mathbb{R} \), we know that \( \mathbb{N} \) is not bounded above, so we can choose \( n \in \mathbb{N} \) such that \( n > 1/(b-a) \). Then \( 1/n < b-a \). We let

\[
S = \{ m \in \mathbb{N} : m/n > a \},
\]

a set which is nonempty because of the Archimedean property once again.

(There exists \( m \in \mathbb{N} \) such that \( m > na \).) Since \( \mathbb{N} \) is well-ordered, \( S \) contains a least element \( m \), which must be \( \geq 2 \). Then

\[
\frac{m-1}{n} \leq a \quad \Rightarrow \quad \frac{m}{n} - \frac{1}{n} \leq a \quad \Rightarrow \quad \frac{m}{n} \leq a + \frac{1}{n} < b.
\]

Hence \( r = m/n \) is a rational number such that \( r \in (a, b) \).

Suppose now that \( \mathbb{R} - \mathbb{Q} \) were open. Then it would contain a nonempty open interval \((a, b)\) in contradiction to the previous proposition. We conclude that \( \mathbb{R} - \mathbb{Q} \) cannot be open.

**Definition.** Suppose \( S \subseteq \mathbb{R} \). Then a point \( x \in \mathbb{R} \) is an *accumulation point* of \( S \) if for every \( \epsilon > 0 \), \( N^*(x; \epsilon) \cap S \neq \emptyset \).

**Proposition 1.5.** If \( x \in \mathbb{R} \) is an accumulation point of \( S \subseteq \mathbb{R} \), then for every \( \epsilon > 0 \), \( N^*(x; \epsilon) \cap S \) is infinite.

Proof: If \( x \) is an accumulation point of \( S \), there exists a point \( x_1 \in N^*(x; \epsilon) \cap S \). Let \( \epsilon_1 = |x_1 - x| \). Then there exists a point \( x_2 \in N^*(x; \epsilon_1) \cap S \subseteq N^*(x; \epsilon) \cap S \), and we let \( \epsilon_2 = |x_2 - x| \). Continuing in this fashion, we get an infinite collection of distinct points \( \{x_1, x_2, x_3, \ldots\} \) in \( N^*(x; \epsilon) \cap S \).

We let \( S' \) denote the set of accumulation points of \( S \). For example, if \( S = \{1, 1/2, 1/3, 1/4, \ldots\} \), the only accumulation point of \( S \) is the point 0. In this
case, $S' = \{0\}$. The notion of accumulation point is the first step on the road to a rigorous definition of limit, as we will see later.

**Proposition 1.6.** A subset $S$ of $\mathbb{R}$ is closed if and only if $S' \subseteq S$.

**Proof:** $\Rightarrow$: Suppose that $S$ is closed and $x \in S'$. If $x \in \mathbb{R} - S$, then $N(x; \epsilon) \subseteq \mathbb{R} - S$ for some $\epsilon > 0$. But then $x \notin S'$.

$\Leftarrow$: Suppose that $S' \subseteq S$. If $x \in \mathbb{R} - S$, then $x \notin S$ and hence $x \notin S'$, so $N^*(x; \epsilon) \cap S = \emptyset$ for some $\epsilon > 0$. But then $N(x; \epsilon) \subseteq \mathbb{R} - S$ for some $\epsilon > 0$. Thus $\mathbb{R} - S$ is open and $S$ is closed.

An important example: the Cantor set. We start with $C_0 = [0, 1]$. We then let

$$C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right] = \frac{C_0}{3} \cup \left(\frac{2}{3} + \frac{C_0}{3}\right),$$

$$C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right] = \frac{C_1}{3} \cup \left(\frac{2}{3} + \frac{C_1}{3}\right),$$

and so forth. Thus for each $n \in \mathbb{N}$, we let

$$C_n = \frac{C_{n-1}}{3} \cup \left(\frac{2}{3} + \frac{C_{n-1}}{3}\right).$$

Note that all of the sets $C_n$ are closed. The intersection

$$C = \bigcap\{C_n : n \in \mathbb{N}\}$$

is also closed, and called the Cantor set.

How long is the Cantor set? To answer this question, we carry out the calculation:

$$\text{Length of } ([0, 1] - C) = \frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \cdots$$

$$= \frac{1}{3} \sum_{n=1}^{\infty} \frac{2^n}{3^n} = \frac{1}{3} \left(\frac{1}{1 - (2/3)}\right) = 1.$$

Since the unit interval $[0, 1]$ has length one, we see that the Cantor set must have length (or measure) zero. The Cantor set also enjoys many other interesting properties. For example, one can also prove that the Cantor set has uncountably many points; see Exercise 14.11 in [1]. it can also be shown that every point of the Cantor set is an accumulation point, and hence $C' = C$; a proof of this latter fact can be found on pages 36,37 of [3].

### 2 Compactness

Suppose that $S$ is a subset of $\mathbb{R}$. We say that a collection $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ is an open cover of $S$ if
1. each $U_\alpha$ is open, and
2. $S \subseteq \bigcup \{U_\alpha : \alpha \in A\}$.

**Definition.** A subset $K \subseteq \mathbb{R}$ is **compact** if whenever $\{U_\alpha : \alpha \in A\}$ is an open cover of $K$, then

$$K \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_m},$$

for some finite subset $\{\alpha_1, \ldots, \alpha_m\} \subseteq A$.

In other words, $K$ is compact if any open cover of $K$ has a finite subcover. Here are some examples of compact and noncompact sets.

**Example 1.** It is easy to see that any finite subset of $\mathbb{R}$ is compact.

**Example 2.** For a more interesting example, we consider the set

$$S = \{0\} \cup \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}.$$

Suppose that we choose an arbitrary open cover $\{U_\alpha : \alpha \in A\}$ of $S$. Then there is some element $U_{\alpha_0}$ in the cover such that $0 \in U_{\alpha_0}$ and there exists $\epsilon > 0$ such that $(-\epsilon, \epsilon) \subseteq U_{\alpha_0}$. By the Archimedean property of $\mathbb{R}$ there is an integer $N \in \mathbb{N}$ such that $N > 1/\epsilon$, and then

$$n > N \implies \frac{1}{n} < \epsilon \implies \frac{1}{n} \in (-\epsilon, \epsilon) \subseteq U_{\alpha_0}.$$

Choose $\alpha_n$ so that $1/n \in U_{\alpha_n}$ for $1 \leq n \leq N$. Then

$$S \subseteq U_{\alpha_0} \cup U_{\alpha_1} \cup \cdots \cup U_{\alpha_N}.$$

Thus $S$ is contained in a finite subcover of the open cover that we chose. This shows that $S$ is indeed compact.

**Example 3.** On the other hand, the set $\mathbb{N}$ of natural numbers is not compact, because $\{(0, n) : n \in \mathbb{N}\}$ is an open cover of $\mathbb{N}$, but it has no finite subcover. Indeed, if

$$\mathbb{N} \subseteq (0, n_1) \cup (0, n_2) \cup \cdots \cup (0, n_m)$$

for some finite $m$, then we could let $N = \max(n_1, \ldots, n_m)$. Since $(0, n_i) \subseteq (0, N)$, we would then have $\mathbb{N} \subseteq (0, N)$ but this would contradict the fact that $\mathbb{N}$ is unbounded (which follows from the Archimedean property of the real numbers).

**Example 4.** Similarly, the “half-open” interval $(0, 1] = \{x \in \mathbb{R} : 0 < x \leq 1\}$ is not compact. Indeed if we let

$$U_n = \left( \frac{1}{n}, 2 \right),$$

then $(0, 1] \subseteq \bigcup \{U_n : n \in \mathbb{N}\}$. 
Note that $n < m \Rightarrow U_n \subseteq U_m$. Thus if it were the case that
\[(0, 1] \subseteq U_{n_1} \cup \cdots \cup U_{n_m},\]
it would follow that
\[(0, 1] \subset U_N, \quad \text{where } N = \max(n_1, \ldots, n_m).\]

But by the Archemedean Property of $\mathbb{R}$, there exists an element $m \in \mathbb{N}$ such that $0 < 1/m < 1/N$ and hence it is not the case that $(0, 1] \subseteq ((1/N), 2)$.

A subset $K \subseteq \mathbb{R}$ is said to be bounded if $K \subseteq N(x; \mathbb{R})$ for some $R > 0$.

**Proposition 2.1.** If a subset of $\mathbb{R}$ is compact, it is closed and bounded.

Proof: Suppose that $K$ is a compact subset of $\mathbb{R}$. For $n \in \mathbb{N}$, the set $U_n = (-n, n)$ is open. Moreover,
\[
\bigcup \{U_n : n \in \mathbb{N}\} = \mathbb{R}.
\]
Thus $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ is an open cover of $K$. Since $K$ is compact, it must have a finite subcover
\[
\{U_{n_1}, \ldots, U_{n_m}\}, \quad \text{so } K \subseteq U_{n_1} \cup \cdots \cup U_{n_k}.
\]
Let $N = \max(n_1, \ldots, n_m)$. Then $K \subset (-N, N)$ and hence $K$ is bounded.

Suppose now that $K$ is compact and $x \in \text{bd}(K) - K$. Let
\[
U_n = \left( -\infty, x - \frac{1}{n} \right) \cup \left( x + \frac{1}{n}, \infty \right). \quad \text{Then } K \subset \bigcup \{U_n : n \in \mathbb{N}\} = \mathbb{R} - \{x\}.
\]
Thus $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ is an open cover of $K$. Since $K$ is compact, it must have a finite subcover
\[
\{U_{n_1}, \ldots, U_{n_m}\}, \quad \text{so } K \subseteq U_{n_1} \cup \cdots \cup U_{n_k}.
\]
Let $N = \max(n_1, \ldots, n_m)$. Then
\[
K \subset \left( -\infty, x - \frac{1}{N} \right) \cup \left( x + \frac{1}{N}, \infty \right)
\]
and hence
\[(x - \epsilon, x + \epsilon) \subseteq \mathbb{R} - K, \quad \text{for } \epsilon \leq \frac{1}{N}.
\]
This contradicts the assumption that $x \in \text{bd}(K)$. Hence $\text{bd}(K) \subseteq K$, and $K$ is closed.

**Proposition 2.2.** Any nonempty finite closed interval $[a, b]$ is compact.

The argument (following [3]) is based upon:
Nesting Lemma. Suppose that \( \{ I_n : n \in \mathbb{N} \} \) is a decreasing nested sequence of nonempty closed intervals in \( \mathbb{R} \),

\[
I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq I_{n+1} \supseteq \cdots
\]

Then \( \cap \{ I_n : n \in \mathbb{N} \} \) is nonempty.

Proof of Lemma: Suppose that \( I_n = [a_n, b_n] \) and let \( S = \{ a_n : n \in \mathbb{N} \} \). \( S \) is nonempty and since \( a_n \leq b_1 \), it is bounded above. Thus \( S \) has a least upper bound \( x \). Since \( x \) is an upper bound, \( a_n \leq x \) for all \( n \). Since \( x \) is a least upper bound and

\[
a_n \leq a_{n+k} \leq b_{n+k} \leq b_n,
\]

for all \( n \). Thus \( x \in [a_n, b_n] \), for every \( n \), and \( x \) lies in the intersection.

Proof of Proposition 2.2: We assume that \( U = \{ U_\alpha : \alpha \in A \} \) is an open cover of \([a, b]\) with no finite subcover. We can divide the interval \( I_1 = [a, b] \) into two equal intervals

\[
[a, b] = \left[ a, \frac{a + b}{2} \right] \cup \left[ \frac{a + b}{2}, b \right].
\]

At least one of the two subintervals cannot be covered by a finite subcollection of \( U \); otherwise the original interval could be so covered. Let \( I_2 = [a_2, b_2] \) be one of the two subintervals, chosen so that it cannot be covered by a finite subcollection of \( U \). We divide \( I_2 \) into two intervals of equal length

\[
I_2 = [a_2, b_2] = \left[ a_2, \frac{a_2 + b_2}{2} \right] \cup \left[ \frac{a_2 + b_2}{2}, b_2 \right].
\]

Once again, at least one of the two subintervals, say \( I_3 \), cannot be covered by a finite subcollection of \( U \). We continue in a similar fashion, thereby obtaining a decreasing nested sequence of nonempty intervals

\[
I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq I_{n+1} \supseteq \cdots \text{ with } \text{Length}(I_{n+1}) = \frac{1}{2}\text{Length}(I_n)
\]

such that no \( I_n \) is covered by a finite subcover of \( U \). By the Lemma, \( \cap \{ I_n : n \in \mathbb{N} \} \) contains some element \( x \). Let \( U_\alpha \) be an element of \( U \) which contains \( x \). Since \( U_\alpha \) is open there exists \( \epsilon > 0 \) such that \( (x - \epsilon, x + \epsilon) \subseteq U_\alpha \). But then

\[
I_n \not\subseteq (x - \epsilon, x + \epsilon) \subseteq U_\alpha
\]

for \( n \) sufficiently large. This contradicts the fact that \( I_n \) was not covered by a finite subcover of \( U = \{ U_\alpha : \alpha \in A \} \).

Proposition 2.3. If \( S \) is a closed subset of a compact set \( K \), then \( S \) is also compact.

Proof: Let \( U = \{ U_\alpha : \alpha \in A \} \) be an open cover of \( S \). Then \( \{ U_\alpha : \alpha \in A \} \cup \{ \mathbb{R} - S \} \) is an open cover of \( K \). It must have a finite subcover

\[
\{ U_{\alpha_1}, \ldots, U_{\alpha_k}, \mathbb{R} - S \}.
\]
But then \( \{U_{\alpha_1}, \ldots, U_{\alpha_k}\} \) is a finite subset of \( \mathcal{U} \) which covers \( S \). Hence \( S \) is compact.

Putting the last three propositions together gives us the key theorem regarding compactness:

**Heine-Borel Theorem.** A subset of \( \mathbb{R} \) is compact if and only if it is closed and bounded.

Proof: The implication \( \Rightarrow \) follows from Proposition 2.1. Conversely, if \( K \) a closed bounded subset of \( \mathbb{R} \), then \( K \) is a closed subset of \([-N,N]\) for some \( N \in \mathbb{N} \). Proposition 2.2 implies that \([-N,N]\) is compact, and Proposition 2.3 implies that \( K \) is compact.

As a corollary, we get:

**Bolzano-Weierstrass Theorem.** Any bounded infinite subset of \( \mathbb{R} \) must have an accumulation point.

Proof: Suppose that \( S \) is an infinite subset of \( \mathbb{R} \) which has no accumulation point. Then since \( S \) contains all its accumulation points, it must be closed. Since it also bounded, it must also be compact. For each \( x \in S \) there is an open neighborhood \( N(x;\epsilon_x) \) of \( x \) such that \( N(x;\epsilon_x) \cap S = \{x\} \). Thus we have an open cover \( \{N(x;\epsilon_x) : x \in S\} \) of \( S \). Since \( S \) is compact, this cover must have a finite subcover,

\[
S \subseteq N(x_1;\epsilon_1) \cup N(x_2;\epsilon_2) \cup \cdots \cup N(x_m;\epsilon_m),
\]

for some \( m \in \mathbb{N} \). But then we would have \( S = \{x_1, \ldots, x_m\} \), a finite set. This gives a contradiction, so the infinite set \( S \) must indeed have an accumulation point, as claimed.

3 The dot product

Ultimately, we would like to develop calculus for functions of several variables. This requires that the theorems we have just proven be extended to the space \( \mathbb{R}^n \) of \( n \)-vectors. We assume that the reader is familiar with vector addition and multiplication by a scalar, from previous courses in calculus.

If \( \mathbf{x} = (x_1, \ldots, x_n) \) and \( \mathbf{y} = (y_1, \ldots, y_n) \) are elements of \( \mathbb{R}^n \), we define their **dot product** by

\[
\mathbf{x} \cdot \mathbf{y} = x_1y_1 + \cdots + x_ny_n.
\]

The dot product satisfies several key axioms:

1. it is symmetric: \( \mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x} \);
2. it is bilinear: \( (a\mathbf{x} + \mathbf{x}') \cdot \mathbf{y} = a(\mathbf{x} \cdot \mathbf{y}) + \mathbf{x}' \cdot \mathbf{y} \);
3. and it is positive-definite: \( x \cdot x \geq 0 \) and \( x \cdot x = 0 \) if and only if \( x = 0 \).

We define the \textit{length} of \( x \) by
\[
| x | = \sqrt{x \cdot x}.
\]

Note that the length of \( x \) is always \( \geq 0 \).

\textbf{Cauchy-Schwarz Theorem.} \textit{The dot product satisfies}
\[
-1 \leq \frac{x \cdot y}{|x||y|} \leq 1. \tag{1}
\]

\textbf{Sketch of proof:} Expand the inequality
\[
0 \leq (x(y \cdot y) - y(x \cdot y)) \cdot (x(y \cdot y) - y(x \cdot y))
\]
and simplify to obtain
\[
0 \leq (x \cdot x)(y \cdot y) - 2(x \cdot y)^2(y \cdot y) + (y \cdot y)(x \cdot y)^2,
\]
which simplifies to
\[
(x \cdot y)^2(y \cdot y) \leq (x \cdot x)(y \cdot y)^2,
\]
or
\[
(x \cdot y)^2 \leq |x|^2|y|^2.
\]

Taking the square root yields the Cauchy-Schwarz inequality (1).

The importance of the Cauchy-Schwarz inequality is that it allows us to define angles between vectors \( x \) and \( y \) in \( \mathbb{R}^n \). Given an number \( t \in [-1, 1] \), there is a unique angle \( \theta \) such that
\[
\theta \in [0, \pi] \quad \text{and} \quad \cos \theta = t.
\]

Thus we can define the angle between two nonzero vectors \( x \) and \( y \) in \( \mathbb{R}^n \) by requiring that
\[
\theta \in [0, \pi] \quad \text{and} \quad \cos \theta = \frac{x \cdot y}{|x||y|}.
\]

Thus we can say that two vectors vectors \( x \) and \( y \) in \( \mathbb{R}^n \) are \textit{perpendicular} or \textit{orthogonal} if \( x \cdot y = 0 \). This provides much of the intuition for dealing with vectors in \( \mathbb{R}^n \).

\textbf{Corollary of Cauchy-Schwarz Theorem.} \textit{If} \( u, v \in \mathbb{R}^n \), \textit{then}
\[
|u + v| \leq |u| + |v|.
\]

\textbf{Proof:} It suffices to check that
\[
|u + v|^2 \leq (|u| + |v|)^2
\]
or
\[
|u|^2 + 2u \cdot v + |v|^2 \leq |u|^2 + 2|u||v| + |v|^2.
\]

But this follows immediately from the Cauchy-Schwarz inequality.
4 Topology of metric spaces

It is when considering $\mathbb{R}^n$ and the various generalizations of $\mathbb{R}^n$ that the power of our earlier definitions of open set and compactness begin to become apparent. Indeed, many of the earlier proofs can be extended to metric spaces with very little additional work.

Definition 4.1. A metric space is a set $X$ together with a function $d : X \times X \to \mathbb{R}$ such that

1. $d(x, y) \geq 0$ and $d(x, y) = 0 \iff x = y$,
2. $d(x, y) = d(y, x)$, and
3. $d(x, z) \leq d(x, y) + d(y, z)$.

The last of these conditions is known as the triangle inequality.

Example 1. The most basic example is $X = \mathbb{R}$ with $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $d(x, y) = |x - y|$.

Example 2. The example needed for multivariable calculus is $X = \mathbb{R}^n$ with $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by $d(x, y) = |x - y|$, the length of $x - y$ being defined as in the preceding section. We call this metric the standard metric on $\mathbb{R}^n$. In verifying that this really is a metric space, the only difficulty is checking the triangle inequality. But this follows from the Corollary in the preceding section when $u = x - y$ and $v = y - z$. Of course, this example includes the previous one as a special case.

Example 3. Let $S$ be a subset of $\mathbb{R}^n$. In this case, we define a distance function $d : S \times S \to \mathbb{R}$ by letting $d(x, y)$ be the distance from $x$ to $y$ in $\mathbb{R}^n$. This provides numerous additional examples of metric spaces.

Further examples can be found in Lay [1], §15.

If $(X, d)$ is a metric space and $\epsilon > 0$ is given, then we define the $\epsilon$-neighborhood $N(x; \epsilon)$ of a point $x \in X$ to be

$$N(x; \epsilon) = \{y \in X : d(x, y) < \epsilon\},$$

and we define the deleted $\epsilon$-neighborhood $N^*(x; \epsilon)$ of $x$ to be

$$N^*(x; \epsilon) = \{y \in X : d(x, y) < \epsilon, y \neq x\},$$

just as we did when $X = \mathbb{R}$. 

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Definition 4.2. A subset $U$ of $X$ is said to be open if

$$x \in U \quad \Rightarrow \quad N(x; \epsilon) \subset U,$$

for some $\epsilon > 0$.

A subset $C$ of $X$ is closed if its complement $X - C$ is open. A subset $K \subseteq X$ is bounded if $K \subseteq N(x; R)$ for some $R > 0$.

Examples of open and closed sets: In any metric space $(X, d)$, the set

$$N(x; R) = \{y \in X : d(x, y) < R\}$$

is open whenever $R > 0$. To see this it suffices to show that

$$y \in N(x; R) \quad \Rightarrow \quad N(y; R - d(x, y)) \subset N(x; R).$$

But

$$z \in N(y; R - d(x, y)) \quad \Rightarrow \quad d(y, z) < R - d(x, y)$$

$$\Rightarrow \quad d(x, z) < d(x, y) + d(y, z) < R$$

$$\Rightarrow \quad d(x, z) < R \quad \Rightarrow \quad z \in N(x; R).$$

It follows that the set

$$X - N(x; R) = \{y \in X : d(x, y) \geq R\}$$

is closed.

By the argument of the preceding paragraph, the open ball of radius $R$ about any given point $(x_0, y_0) \in \mathbb{R}^2$,

$$N((x_0, y_0); R) = \{(x, y) \in \mathbb{R}^2 : (x - x_0)^2 + (y - y_0)^2 < R^2\},$$

is an open subset of $\mathbb{R}^2$. However, it is not a closed subset of $\mathbb{R}^2$ because the point $(x_0 + R, y_0)$ lies in $\mathbb{R}^2 - N((x_0, y_0); R)$, but there is no $\epsilon$-neighborhood about $(x_0 + R, y_0)$ which is entirely contained in $\mathbb{R}^2 - N((x_0, y_0); R)$. Similarly, one can show that the closed ball

$$B((x_0, y_0); R) = \{(x, y) \in \mathbb{R}^2 : (x - x_0)^2 + (y - y_0)^2 \leq R^2\}$$

is closed but not open.

If $(a, b)$ and $(c, d)$ are nonempty open intervals, the Cartesian product $(a, b) \times (c, d)$ is open but not closed. If $[a, b]$ and $[c, d]$ are nonempty closed intervals, $[a, b] \times [c, d]$ is closed but not open.

Much of the theory of open and closed sets in $\mathbb{R}$ carries over to open and closed sets in an arbitrary metric space $(X, d)$ with very little change:

**Proposition 4.1.** If $(X, d)$ is any metric space, the empty set $\emptyset$ and the whole space $X$ are open. The union of an arbitrary collection of open sets is open. The intersection of a finite collection of open sets is open.
The proof is virtually the same as the proof given in §1.

**Proposition 4.2.** If \((X, d)\) is any metric space, the empty set \(\emptyset\) and the whole space \(X\) are closed. The intersection of an arbitrary collection of closed sets is closed. The union of a finite collection of closed sets is closed.

Once again, the proof is virtually the same as that given in §1.

The metric \(d\) of a metric space \((X, d)\) can be restricted to any subset \(S \subseteq X\), making \(S\) into a metric space in its own right. This gives rise to the notion of an open subset of \(S\). (Note that for \(x \in S\), the \(\epsilon\)-neighborhoods of \(x\) in \(S\) are just the intersections \(N(x; \epsilon) \cap S\), where \(N(x; \epsilon)\) is an \(\epsilon\)-neighborhood of \(x\) in \(X\).)

**Proposition 4.3.** Let \((X, d)\) be a metric space and let \(S\) be a subset of \(X\). Then a subset \(U\) of \(S\) is open if and only if \(U = V \cap S\), where \(V\) is open in \(X\).

**Proof:** Suppose that \(V\) is an open subset of \(X\) and \(U = V \cap S\). If \(x \in V \cap S\), then since \(V\) is open in \(X\), there exists an \(\epsilon > 0\) such that \(N(x; \epsilon) \subseteq V\). But then \(N(x; \epsilon) \cap S \subseteq V \cap S = U\). So \(U\) is open in \(S\).

Conversely, if \(U\) is an open subset of \(S\), then for \(x \in U \cap S\), there exists an \(\epsilon_x > 0\) such that \(N(x; \epsilon_x) \cap S \subseteq U\). We let

\[ V = \bigcup \{ N(x; \epsilon_x) : x \in U \cap S \}. \]

Then \(V\) is an open subset of \(X\) and \(V \cap S = U\).

**Definition 4.3.** We say that a subset \(K\) of a metric space \((X, d)\) is compact if every open cover of \(K\) has a finite subcover.

### 5 The Heine-Borel Theorem for \(\mathbb{R}^n\)

Unfortunately, the Heine-Borel Theorem that we proved for \(\mathbb{R}\) in §2 cannot be extended to all metric spaces. (Compact subsets of an arbitrary metric space are closed and bounded, but the converse is NOT true.) However, the Heine-Borel Theorem can be extended for the metric space \((\mathbb{R}^n, d)\) that is used in the calculus of several variables:

**Heine-Borel Theorem for \(\mathbb{R}^n\).** A subset of \(\mathbb{R}^n\) with the standard metric is compact if and only if it is closed and bounded.

The proof we gave before actually extends from \(\mathbb{R}\) to \(\mathbb{R}^n\). First we need to show that a compact subset \(K\) of \(\mathbb{R}^n\) is bounded. If not we can construct a collection of open sets,

\[ \mathcal{U} = \{ U_n : n \in \mathbb{N} \}, \quad U_n = N(0; n), \]

which covers \(K\) and has no finite subcover.

Next we must show that a compact subset \(K\) of \(\mathbb{R}^n\) is closed. If \(K\) is not closed there exists \(x \in \mathbb{R}^n - K\) which lies in \(K'\). In this case, we can construct...
a collection of open sets, 
\[ \mathcal{U} = \{U_n : n \in \mathbb{Z}\}, \quad U_n = N(x; 2^n), \]
which covers \( K \) but has no finite subcover.

Finally, we must show that if \( K \) is a closed bounded subset of \( \mathbb{R}^n \), \( K \) is compact. For this we need a modification of our earlier nesting lemma. We say that \( I \subset \mathbb{R}^n \) is a closed \( n \)-cube if
\[ I = [a_1, b_1] \times \cdots \times [a_n, b_n] \]
with \( b_1 - a_1 = \cdots = b_n - a_n \).

We say that the length of the closed \( n \)-cube is \( b_1 - a_1 \).

**Nesting Lemma.** Suppose that \( \{I_k : k \in \mathbb{N}\} \) is a decreasing nested sequence of nonempty closed \( n \)-cubes in \( \mathbb{R}^n \),
\[ I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq I_{n+1} \supseteq \cdots \]
Then \( \bigcap \{I_n : n \in \mathbb{N}\} \) is nonempty.

**Proof of Lemma:** Suppose that
\[ I_k = [a_{k,1}, b_{k,1}] \times \cdots \times [a_{k,n}, b_{k,n}] \]
Then for each \( j, 1 \leq j \leq n \),
\[ [a_{1,j}, b_{1,j}] \supseteq [a_{1,j}, b_{1,j}] \supseteq \cdots \supseteq [a_{1,j}, b_{1,j}] \supseteq \cdots \]
is a nested sequence of intervals. By the earlier nesting lemma, there exists a point
\[ x_j \in \bigcap \{[a_{k,j}, b_{k,j}] : k \in \mathbb{N}\}. \]
Then the point \( (x_1, \ldots, x_n) \) lies in \( \bigcap \{I_n : n \in \mathbb{N}\} \).

Once we have this Nesting Lemma, we can prove that for any \( M > 0 \), the closed unit \( n \)-cube \( I_1 = [-M, M] \times \cdots \times [-M, M] \) of length \( 2M \) is compact by a modification of the argument we gave before. Suppose that \( \mathcal{U} = \{U_\alpha : \alpha \in A\} \) is an open cover of \( I_1 \) with no finite subcover. We can divide \( I_1 \) by the coordinate hyperplanes into \( 2^n \) closed cubes of length \( M \). At least one of these cubes \( I_2 \) cannot be covered by a finite subcollection of \( \mathcal{U} \); otherwise the original interval could be so covered. We then divide \( I_2 \) into \( 2^n \) closed cubes of length \( M/2 \). Once again, at least one of these cubes, say \( I_3 \) cannot be covered by a finite subcollection of \( \mathcal{U} \). We continue in a similar fashion, thereby obtaining a decreasing nested sequence
\[ I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq I_{n+1} \supseteq \cdots \]
with \( \text{Length}(I_{n+1}) = \frac{1}{2} \text{Length}(I_n) \)
such that no \( I_n \) is covered by a finite subcover of \( \mathcal{U} \). By the new Nesting Lemma, \( \bigcap \{I_n : n \in \mathbb{N}\} \) contains some element \( (x_1, \ldots, x_n) \). Let \( U_\alpha \) be an element of \( \mathcal{U} \) which contains \( x \). Since \( U_\alpha \) is open there exists \( \epsilon > 0 \) such that
\[ (x_1 - \epsilon, x_1 + \epsilon) \times \cdots \times (x_n - \epsilon, x_n + \epsilon) \subseteq U_\alpha. \]
But then
\[ I_n \subseteq (x_1 - \epsilon, x_1 + \epsilon) \times \cdots \times (x_n - \epsilon, x_n + \epsilon) \subseteq U_\alpha \]
for \( n \) sufficiently large. This contradicts the fact that \( I_n \) was not covered by a finite subcover of \( \mathcal{U} = \{ U_\alpha : \alpha \in A \} \). Hence the \( n \)-cube \( I_1 = [-M, M] \times \cdots \times [-M, M] \) is compact.

Finally, if \( K \) is any closed bounded subset of \( \mathbb{R}^n \), then \( K \) is a closed subset of \( [-M, M] \times \cdots \times [-M, M] \) for some \( M > 0 \). The proof of Proposition 2.3 now shows that \( K \) is compact.

**Bolzano-Weierstrass Theorem for \( \mathbb{R}^n \).** Any bounded infinite subset of \( \mathbb{R}^n \) must have an accumulation point.

The proof is virtually identical to the proof of the Bolzano-Weierstrass Theorem for \( \mathbb{R} \) presented in §2, once we have the Heine-Borel Theorem for \( \mathbb{R}^n \).

**References**

