I. Real roots.

(If you took Math 3CI, you will have seen an earlier version of this project.) We consider a cart moving along a track attached to a wall by means of a spring, and let

\[ x(t) = \text{position of the cart to the right of equilibrium at time } t. \]

We consider three forces acting on the cart. First, a spring force

\[ F_{\text{spring}} = -kx, \]

where \( k \) is a positive constant, called the spring constant. Second, a linear damping force,

\[ F_{\text{damping}} = -c \frac{dx}{dt}, \]

where \( c \) is a positive constant, the damping force being assumed to be linear in the velocity and in the opposite direction to the motion of the cart. Finally, we suppose that there is an external force

\[ F_{\text{external}} = f(t), \]

where \( f(t) \) is a function that might be given. We would like to determine the motion of the cart.

In accordance with Newton’s law of motion, the total force satisfies the equation

\[ F_{\text{total}} = m \frac{d^2x}{dt^2}, \]

where \( m \) is the mass of the cart, and hence the function \( x(t) \) must satisfy

\[ m \frac{d^2x}{dt^2} = -c \frac{dx}{dt} - kx + f(t). \]
This is a second order linear differential equation with constant coefficients:
\[
\frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx = f(t),
\]
where the constants \(m, c,\) and \(k\) are positive.

Linear differential equation can be written in terms of linear operators. A linear operator \(L\) takes a well-behaved function \(x(t)\) to a new function \(L(x)(t)\) in such a way that
\[
L(c_1x_1 + c_2x_2) = c_1L(x_1) + c_2L(x_2),
\]
whenever \(c_1\) and \(c_2\) are constants. An example of a linear operator is the ordinary differentiation operator \(D = d/dt\)—it is a familiar fact from first-year calculus that this operator satisfies (2):
\[
D(c_1x_1 + c_2x_2) = c_1D(x_1) + c_2D(x_2).
\]
We will use \(D^2\) to denote the linear operator which differentiates twice in succession; thus \(D^2(x) = (d^2x/dt^2)\). More linear operators can be constructed from these—for example,
\[
L = 3D^2 + 5D + 4, \quad L(x) = 3\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 4x.
\]
The key point about linear operators is that if \(L\) is a linear operator, the solutions to the equation \(L(x) = 0\) must satisfy the principle of superposition, because by (2),
\[
L(x_1) = 0 \text{ and } L(x_2) = 0 \Rightarrow L(c_1x_1 + c_2x_2) = 0.
\]
Thus, if we write \(L = mD^2 + cD + k\), then the differential equation (1) can be rewritten as
\[
L(x)(t) = f(t).
\]
The equation (1) is said to be homogeneous if \(f(t)\) is zero. In this case, we can usually find the general solution as follows:

**Step I.** Assume that the solution has the special form
\[
x = e^{\lambda t},
\]
where \(\lambda\) is a constant, and solve for \(\lambda\). This usually gives two linearly independent solutions \(e^{\lambda_1 t}\) and \(e^{\lambda_2 t}\) to the equation.

**Step II.** Use the superposition principle to write down the general solution to the differential equation:
\[
x(t) = c_1e^{\lambda_1 t} + c_2e^{\lambda_2 t}.
\]

1. a. Find the general solution to the equation for the motion of the cart in the case where \(m = 1, c = 5\) and \(k = 6\).

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b. Find the particular solution to the above equation which satisfies the initial conditions
\[ x(0) = 1, \quad \frac{dx}{dt}(0) = 0. \]

2. a. Find the general solution to the differential equation
\[ 4 \frac{d^2x}{dt^2} + 8 \frac{dx}{dt} + 3x = 0. \]

b. Find the particular solution which satisfies the initial conditions
\[ x(0) = 2, \quad \frac{dx}{dt}(0) = 5. \]

We next consider nonhomogeneous linear differential equations
\[ m \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx = f(t), \quad (4) \]
where \( f(t) \) is nonzero. There is a two-step method for solving equations of this type:

**Step I.** Find the general solution to the associated homogeneous differential equation
\[ a \frac{d^2x}{dt^2} + b \frac{dx}{dt} + cx = 0, \quad (5) \]
by the method we have discussed in the preceding sections.

**Step II.** Find a particular solution to the original nonhomogeneous equation (4). The general solution to (4) is the sum of the general solution to (5) and the particular solution to (4).

The linear operator notation makes it clear why this method works. Indeed, if we let \( L \) denote the linear operator,
\[ L = m D^2 + c D + k = \left( m \frac{d^2}{dt^2} + c \frac{d}{dt} + k \right), \]
the differential equation (4) can be expressed as \( L(x) = f \), and the associated homogeneous equation as \( L(x) = 0 \). The linearity of \( L \),
\[ L(c_1 x_1 + c_2 x_2) = c_1 L(x_1) + c_2 L(x_2), \]
implies that if \( x_1 \) and \( x_2 \) are solutions to the associated homogeneous equation \( L(x) = 0 \), then so is \( c_1 x_1 + c_2 x_2 \) for any choice of the constants \( c_1 \) and \( c_2 \). Moreover, if \( x = c_1 x_1 + c_2 x_2 \) is the general solution to \( L(x) = 0 \) and \( x_p \) is a particular solution to \( L(x) = f \), then
\[ L(c_1 x_1 + c_2 x_2 + x_p) = L(c_1 x_1 + c_2 x_2) + L(x_p) = 0 + f = f, \]
so for each choice of constants \( c_1 \) and \( c_2 \), the function

\[
x = c_1 x_1 + c_2 x_2 + x_p
\]  

(6)
is a solution to \( L(x) = f \). Indeed, (6) is the general solution to \( L(x) = f \).

3. a. Suppose that we want to solve the differential equation

\[
\frac{d^2 x}{dt^2} + 5 \frac{dx}{dt} + 6x = 20e^{2t},
\]  

(7)
which can be written in operator notation as

\[
L(x) = 20e^{2t}, \quad \text{where} \quad L = D^2 + 5D + 6.
\]

The associated homogeneous equation \( L(x) = 0 \) has the general solution

\[
x = c_1 e^{-2t} + c_2 e^{-3t}.
\]

To find a particular solution to the nonhomogeneous equation \( L(x) = 20e^{2t} \), we can employ the method of judicious guessing, also called the method of undetermined coefficients. In this method, we assume that the particular solution to \( L(x) = 20e^{2t} \) is of the form \( x_p = Ae^{2t} \), and determine the coefficient \( A \). Using this idea, find the general solution to (7).

b. Find the particular solution which satisfies the initial conditions

\[
x(0) = 1, \quad \frac{dx}{dt}(0) = 0.
\]

4. Another example is given by the differential equation

\[
\frac{d^2 x}{dt^2} - 5 \frac{dx}{dt} + 4x = \sin t,
\]  

(8)
which can be written in operator notation as

\[
L(x) = \sin t, \quad \text{where} \quad L = D^2 - 5D + 4.
\]

The associated homogeneous equation \( L(x) = 0 \) has the general solution

\[
x = c_1 e^t + c_2 e^{4t}.
\]

To find a particular solution to the nonhomogeneous equation \( L(x) = \sin t \), we can assume that it is of the form \( x_p(t) = A \cos t + B \sin t \), and determine the coefficients \( A \) and \( B \). Using this approach find the general solution to the nonhomogeneous differential equation (8).

b. Find the particular solution to (8) which satisfies the initial conditions

\[
x(0) = 1, \quad \frac{dx}{dt}(0) = 0.
\]
II. Complex roots.

In most of calculus, one deals with real numbers, representable by arbitrary decimal expansions. But in some problems the set $\mathbb{R}$ of real numbers is not sufficiently general; this occurs, for example, whenever we need to take the square root of a negative number. The idea behind complex numbers is to introduce an “imaginary number” $i$ to represent the square root of $-1$. A complex number is a number of the form $x + iy$, where $x$ and $y$ are real numbers. Any nonzero complex number has two square roots.

Complex numbers are often represented by points in the $(x, y)$-plane. We call $x$ the real part and $y$ the imaginary part of the complex number $x + iy$.

Addition and multiplication of complex numbers are defined by the formulae

\[(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2),\]
\[(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1).\]

For example,
\[(3 + 4i) + (2 + 5i) = 5 + 9i, \quad (3 + 4i)(2 - 1) = 6 + 8i - 3i - 4i^2 = 10 + 5i.\]

Addition is just another version of the usual “vector addition” used in navigation, i.e.
\[(3 \text{ miles east} + 4 \text{ miles north}) + (2 \text{ miles east} + 5 \text{ miles north}) = 5 \text{ miles east} + 9 \text{ miles north},\]

but the geometric meaning of multiplication is more complicated, as we will see later. Addition and multiplication of complex numbers satisfy all the usual rules of arithmetic, and indeed, these operations make the set $\mathbb{C}$ of complex numbers into what mathematicians call a “field.”

You can construct complex numbers from real numbers by using matrices. Indeed, you can write the matrix

\[
\begin{pmatrix}
 x & -y \\
 y & x
\end{pmatrix}
\]

for the complex number $x + iy$.

Then the rules for complex addition and multiplication become just the rules for matrix addition and multiplication.

Within the field of complex numbers, subtraction presents no problem, but division requires a little skill. First some terminology; the conjugate of a complex number $x + iy$ is the number $x - iy$ obtained by replacing $i$ by $-i$. The product of a complex number by its conjugate is always a real number, a number whose imaginary part is zero; indeed

\[(x + iy)(x - iy) = x^2 + y^2.\]

To calculate the quotient of two complex numbers, multiply both numerator and denominator by the conjugate of the denominator. For example,
\[
\frac{5 + 2i}{4 + 3i} = \frac{5 + 2i}{4 + 3i} \cdot \frac{4 - 3i}{4 - 3i} = \frac{20 + 6i - 15i + 8i^2}{16 + 9} = \frac{26 - 7i}{25} = \frac{26}{25} - \frac{7}{25}i.
\]
Recall the power series formulae obtained in calculus courses when $x$ is real:

\begin{align*}
e^x &= 1 + x + (1/2!)x^2 + (1/3!)x^3 + (1/4!)x^4 \ldots \\
\sin x &= x - (1/3!)x^3 + (1/5!)x^5 - \ldots \\
\cos x &= 1 - (1/2!)x^2 + (1/4!)x^4 - \ldots \\
\end{align*}

If $z = x + iy$ is a complex number, we can define the functions $e^z$, $\sin z$, and $\cos z$ by the same formulae

\begin{align*}
e^z &= 1 + z + (1/2!)z^2 + (1/3!)z^3 + (1/4!)z^4 + \ldots \\
\sin z &= z - (1/3!)z^3 + (1/5!)z^5 - \ldots \\
\cos z &= 1 - (1/2!)z^2 + (1/4!)z^4 - \ldots \\
\end{align*}

These complex-valued functions satisfy many of the same rules as the original real-valued functions—for example,

\begin{align*}
e^{z+w} &= e^z e^w, \\
\frac{d}{dt}(e^{tz}) &= z e^{tz}. \\
\end{align*}

Moreover,

\begin{align*}
e^{i\theta} &= 1 + i \theta + (1/2!)(i \theta)^2 + (1/3!)(i \theta)^3 + (1/4!)(i \theta)^4 + \ldots \\
&= 1 + i \theta + i^2(1/2!)\theta^2 + i^3(1/3!)\theta^3 + i^4(1/4!)\theta^4 + \ldots \\
&= 1 - (1/2!)\theta^2 + (1/4!)\theta^4 - \ldots + i(\theta - (1/3!)\theta^3 + (1/5!)\theta^5 - \ldots) \\
&= \cos \theta + i \sin \theta \\
\end{align*}

This gives the celebrated formula due to Euler,

\[ e^{i\theta} = \cos \theta + i \sin \theta. \]

For us, the key application of Euler’s formula is to differential equations.

5. a. Use complex numbers and Euler’s formula to find the general solution to the differential equation

\[ \frac{d^2x}{dt^2} + 6 \frac{dx}{dt} + 10x = 0. \quad (9) \]

b. Use Euler’s formula to write the solution in real form, that is, in the form

\[ x = c_1 e^{bt} \cos(\omega t) + c_2 e^{bt} \sin(\omega t), \quad (10) \]

for certain constants $b$ and $\omega$, where $c_1$ and $c_2$ are constants of integration. As $c_1$ and $c_2$ range throughout all real numbers, (10) ranges throughout all the real solutions to (9).

c. Find the solution to the initial value problem

\[ \frac{d^2x}{dt^2} + 6 \frac{dx}{dt} + 10x = 0, \quad x(0) = 1, \quad \frac{dx}{dt}(0) = 0. \]
d. Find the general solution to the nonhomogeneous differential equation

\[ \frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 10x = t^2. \]

Hint: To find a particular solution to the nonhomogeneous differential equation \( L(x) = t^2 \), we assume that it is of the form \( x_p(t) = At^2 + Bt + C \), and determine the coefficients \( A, B, \) and \( C \).

e. Find the solution to the initial value problem

\[ \frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 10x = t^2, \quad x(0) = 1, \quad \frac{dx}{dt}(0) = 0. \]

III. Summary. We have seen how to find solutions to homogeneous linear differential equations with constant coefficients by assuming solutions are superpositions of exponential functions. Moreover, we have seen that the method of judicious guessing is often useful for finding a particular solution to the corresponding nonhomogeneous linear differential equations. The method of judicious guessing doesn’t always work, but when it does, it is usually much faster than other methods, which will be discussed later. The following table gives the most frequently used judicious guesses:

<table>
<thead>
<tr>
<th>For a particular solution to ...</th>
<th>try setting ...</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L(x) = ae^{\lambda t} )</td>
<td>( x_p(t) = Ae^{\lambda t} )</td>
</tr>
<tr>
<td>( L(x) = a \cos t ) or ( b \sin t )</td>
<td>( x_p(t) = A \cos t + B \sin t )</td>
</tr>
<tr>
<td>( L(x) = a_n t^n + a_{n-1} t^{n-1} + \ldots )</td>
<td>( x_p(t) = A_n t^n + A_{n-1} t^{n-1} + \ldots )</td>
</tr>
</tbody>
</table>

Dangerous curve! The judicious guess in the above table will not work when it is a solution to the associated homogeneous equation. In this case, the judicious guess needs to be multiplied by a suitable multiple of \( t \).

Homework I. Due Friday, January 8, 2009.

H.1.1. Find the general solution to each of the following nonhomogeneous differential equations by means of the following steps: First, find the general solution to the associated homogeneous equation. Then use the method of judicious guessing to find a particular solution to the original equation. Finally, add the results together to get the general solution to the nonhomogeneous equation:

a. \( \frac{d^2x}{dt^2} - 5\frac{dx}{dt} + 6x = e^{-t} \).

b. \( (D^2 + 7D + 6)(x) = \cos t \).

H.1.2. Find the solutions to the following initial-value problems:

a. \( \frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 4x = e^{-t}, \quad x(0) = 1, \quad (dx/dt)(0) = 0 \).

b. \( (D^2 - D - 12)(x) = t^2, \quad x(0) = 0, \quad (dx/dt)(0) = 0 \).