Math 5AI: Project 10
Stability of solutions to linear systems of
differential equations

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Definition. A constant solution \((x_0, y_0)\) to the system of differential equations
\[
\begin{align*}
\frac{dx}{dt} &= f(x, y), \\
\frac{dy}{dt} &= g(x, y)
\end{align*}
\]
is said to be asymptotically stable if trajectories which start near \((x_0, y_0)\) approach \((x_0, y_0)\) as \(t \to \infty\). It is stable if it satisfies the weaker condition that trajectories which start near \((x_0, y_0)\) stay near \((x_0, y_0)\) as \(t \to \infty\). It is unstable if it is not stable.

If the system of differential equations is nonlinear, it is usually quite difficult to solve it explicitly. However, it is often quite useful to find the constant solutions to a nonlinear system and determine their stability. Often it is possible to determine whether a given constant solution is stable or not by investigating whether the “closest linear approximation” near the constant solution is stable.

A. Stability of linear systems

1. a. Find the eigenvalues and the corresponding eigenspaces for the linear system
\[
\begin{align*}
\frac{dx}{dt} &= 2x + 5y, \\
\frac{dy}{dt} &= x - 2y.
\end{align*}
\]
b. Find the general solution to this linear system.

c. Sketch the eigenspaces in the \((x, y)\)-plane and put arrows on them to indicate the direction of fluid flow along the eigenspaces. Give a rough sketch of the other orbits in the \((x, y)\)-plane.

d. Is the constant solution \((0, 0)\) for this linear system stable? Is it asymptotically stable?
2. a. Find the eigenvalues and the corresponding eigenspaces for the linear system

\[
\begin{align*}
    \frac{dx}{dt} &= -x + 3y, \\
    \frac{dy}{dt} &= -x - 5y.
\end{align*}
\]

b. Find the general solution to this linear system.

c. Sketch the eigenspaces in the \((x, y)\)-plane and put arrows on them to indicate the direction of fluid flow along the eigenspaces. Give a rough sketch of the other orbits in the \((x, y)\)-plane.

d. Is the constant solution \((0, 0)\) for this linear system stable? Is it asymptotically stable?

3. a. Find the eigenvalues and the corresponding eigenspaces for the linear system

\[
\begin{align*}
    \frac{dx}{dt} &= -2x + y, \\
    \frac{dy}{dt} &= -x - 2y.
\end{align*}
\]

b. Find the general solution to this linear system.

c. Sketch the eigenspaces in the \((x, y)\)-plane and put arrows on them to indicate the direction of fluid flow along the eigenspaces. Give a rough sketch of the other orbits in the \((x, y)\)-plane.

d. Is the constant solution \((0, 0)\) for this linear system stable? Is it asymptotically stable?

4. a. Find the eigenvalues and the corresponding eigenspaces for the linear system

\[
\begin{align*}
    \frac{dx}{dt} &= y, \\
    \frac{dy}{dt} &= -x.
\end{align*}
\]

b. Find the general solution to this linear system.

c. Give a rough sketch of the other orbits in the \((x, y)\)-plane.

d. Is the constant solution \((0, 0)\) for this linear system stable? Is it asymptotically stable?

Remark: Mathematical software available on the web can be used to provide somewhat nicer sketches of the orbits for the preceding problems. To do this use the software package (PPLANE 2005.10) available at:

http://math.rice.edu/~dfield/dfpp.html
By clicking at a point in the “phase plane” window, you can have the software sketch a solution curve which starts at that point. You may want to try this software out over the weekend on the above systems.

5. a. Find the general solution to the linear system

\[
\begin{align*}
\frac{dx}{dt} &= 2(x - 3) - 5(y - 1), \\
\frac{dy}{dt} &= (x - 3) - 2(y - 1).
\end{align*}
\]

Hint: Reduce this to problem 1 by making the change of variables

\[
\bar{x} = x - 3, \quad \bar{y} = y - 1.
\]

b. Give a rough sketch of the paths traced out by the solutions in the \((x,y)\)-plane.

c. Is the constant solution \((3,1)\) for this linear system stable? Is it asymptotically stable?

6. a. Find the constant solution \((x_0,y_0)\) to the linear system

\[
\begin{align*}
\frac{dx}{dt} &= -x + 3y - 4, \\
\frac{dy}{dt} &= -x - 5y + 12.
\end{align*}
\]

b. Find the general solution to this linear system. Hint: Reduce this to problem 2 by making the change of variables

\[
\bar{x} = x - x_0, \quad \bar{y} = y - y_0.
\]

c. Give a rough sketch of the paths traced out by the solutions in the \((x,y)\)-plane.

d. Is the constant solution for this linear system stable? Is it asymptotically stable?

7. Consider dynamical systems of the form

\[
\begin{align*}
\frac{dx}{dt} &= a_{11}x + a_{12}y, \\
\frac{dy}{dt} &= a_{21}x + a_{22}y,
\end{align*}
\]

where the \(a_{ij}\)'s are constants. What conditions on the eigenvalues of

\[
A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}
\]

are needed for the constant solution at \((0,0)\) to be stable? to be asymptotically stable?
B. The predator-prey equations

In Math 3C, you studied the Volterra-Lotka predator-prey equations as an example of nonlinear systems of differential equations that could be studied by qualitative methods. We imagine two species of animals on an island, foxes and rabbits, and let

\[ x(t) = (\text{population of rabbits at time } t), \quad y(t) = (\text{population of foxes at time } t). \]

Under appropriate hypotheses, we can model the populations of these species by a system of differential equations,

\[
\begin{align*}
\frac{dx}{dt} &= (A - By)x, \\
\frac{dy}{dt} &= (-C + Dx)y,
\end{align*}
\]

where \( A, B, C \) and \( D \) are constant. By setting \( dx/dt = 0 \) and \( dy/dt = 0 \), we find that the constant solutions are at the points \((0,0)\) and \((C/D, A/B)\). We would like to study the qualitative behaviour of the solutions to this differential equation near the constant solution \((C/D, A/B)\). We ask the question: What are the trajectories of the solutions that start near \((C/D, A/B)\)?

To answer this question, it is useful to employ a change of variables. Thus we set

\[
\begin{align*}
\bar{x} &= x - (C/D), \\
\bar{y} &= y - (A/B),
\end{align*}
\]

or

\[
\begin{align*}
x &= \bar{x} + (C/D), \\
y &= \bar{y} + (A/B),
\end{align*}
\]

The point of this change of coordinates is that when \((x, y)\) is close to \((C/D, A/B)\), then \((\bar{x}, \bar{y})\) will be close to \((0,0)\).

8. a. Write out the system (1) in terms of the new coordinates \((\bar{x}, \bar{y})\). (If it helps, you can set \( A = B = C = D = 1 \) first, carry through the problem, and then come back to do the case for general \( A, B, C \) and \( D \).)

b. When \((x, y)\) is close to the point \((C/D, A/B)\), \((\bar{x}, \bar{y})\) will be close to \((0,0)\) and terms including \( \bar{x}\bar{y} \) should be very small. You obtain the linearization of the system you found in part a by setting these terms equal to zero. What is the resulting linearization?

c. Find the solution to the linearization you found in part b. Your solution should be oscillatory in nature. What is the period of oscillation? What does this tell you about the behavior of the fox and rabbit populations when they are near the equilibrium population?

We next consider the effect of adding a term \(-\epsilon x^2\) to the first of equations (1), where \( \epsilon \) is a small positive number, the new term representing the effect of overcrowding on the rabbit population:

\[
\begin{align*}
\frac{dx}{dt} &= (A - By - \epsilon x)x, \\
\frac{dy}{dt} &= (-C + Dx)y.
\end{align*}
\]
To analyze the qualitative behaviour of the new system, we first find the constant solutions, the solutions to the system

\[
\begin{align*}
(A - B y - \epsilon x)x &= 0, \\
(-C + D x)y &= 0.
\end{align*}
\]

From the second of these equations, we see that either \( y = 0 \) or \( x = C/D \). If \( y = 0 \) it follows from the first equation that \( (A-\epsilon x)x = 0 \), so either \( x = 0 \) or \( x = A/\epsilon \). If \( x = C/D \), it follows from the first equation that \( (A-B y-\epsilon(C/D))(C/D) = 0 \), from which we conclude that \( y = (A - \epsilon(C/D))/B = (A/B) - \epsilon(C/BD) \). Thus we see that the only constant solutions are

\[(x, y) = (0, 0), (A/\epsilon, 0), \text{ or } (C/D, (A/B) - \epsilon(C/BD)) \]

The third of these is close to the constant solution \((C/D, A/B)\) of (1) when \( \epsilon \) is small. We ask the question: What is the behaviour of trajectories to the new system that start near \((C/D, A/B)\)?

To answer this question, we employ a different change of variables, and set

\[
\begin{align*}
\bar{x} &= x - (C/D), \\
\bar{y} &= y - (A/B) + \epsilon(C/BD),
\end{align*}
\]

or

\[
\begin{align*}
x &= \bar{x} + (C/D), \\
y &= \bar{y} + (A/B) - \epsilon(C/BD),
\end{align*}
\]

9. a. Write out the system (1) in terms of the new coordinates \((\bar{x}, \bar{y})\).

b. Terms including \(\bar{x}\bar{y}\) or \(\bar{x}^2\) should be very small. You obtain the linearization of the system you found in part a by setting these terms equal to zero. What is the resulting linearization?

c. Find the eigenvalues for the linear system you found in part b. Is the solution to the linearization asymptotically stable? What does this suggest about the asymptotic stability of the system (3)?

C. The pendulum equation

We want to describe the motion of a pendulum, so we suppose that a bob of mass \( m \) is attached to the ceiling by a cord of length \( a \), and assume that the bob is free to swing back and forth on a circle \( C \) in a fixed vertical plane. Let \( \theta(t) \) be the directed angle from the equilibrium position to the position of the pendulum at time \( t \); \( F_C \) the component of gravitational force tangent to \( C \). According to Newton’s second law of motion (force = mass times acceleration),

\[ F_C = ma(d^2\theta/dt^2). \]

On the other hand, if \( g \) is the acceleration, we see that

\[ F_C = -mg\sin \theta. \]
Setting the two expressions for $F_C$ equal to each other, we obtain
\[ ma(d^2\theta/dt^2) = -mg\sin\theta. \]
Divide by $ma$ and replace $\theta$ by $x$ to obtain the pendulum equation
\[ (d^2x/dt^2) = -(g/a)\sin x. \] (5)

This nonlinear differential equation cannot be solved completely using only the elementary functions ($e^x$, $\sin x$, etc.) we are already familiar with. Of course, we could define new functions $\phi_{c_0,c_1}$, by requiring that $\phi_{c_0,c_1}(t)$ be the unique solution to the initial value problem
\[ (d^2x/dt^2) = -(g/a)\sin x, \quad x(0) = c_0, \quad (dx/dt)(0) = c_1. \]

Using numerical methods, we could tabulate values for these new functions.

The pendulum equation is equivalent to the first order system
\begin{align*}
\frac{dx}{dt} &= v, \\
\frac{dv}{dt} &= -(g/a)\sin x.
\end{align*} (6)

Our goal is to investigate the qualitative behaviour of this dynamical system.

The first step is to find the equilibria or constant solutions. To do this we must solve the system of equations
\begin{align*}
v &= 0, \\
-(g/a)\sin x &= 0.
\end{align*}

The solutions are
\[ (x,v) = (k\pi,0), \quad \text{where } k \text{ is an integer}. \]

Next, we eliminate $t$ from the two equations (6) to obtain the differential equations for the orbits in the phase plane,
\begin{align*}
avdv + g\sin xdx &= 0, \\
mavdv + mg\sin xdx &= 0,
\end{align*}
which integrates to
\[ E(x,v) = (1/2)mav^2 - mg\cos x = c, \] (7)
where $c$ is a constant of integration. The orbits in the $(x,y)$-plane are simply the level sets of $E$.

10. a. Show that $(0,0)$ is a local minimum for the function $E(x,v)$. From this, can you conclude that $(0,0)$ is stable? asymptotically stable? Is the constant solution $(\pi,0)$ stable?

b. It is interesting to linearize the dynamical system (6) near the constant solution $(0,0)$. Since the usual power series expression for $\sin$—since $\sin x =
\[ x - (1/6)x^3 + \cdots, \] which is closely approximated by \( x \) when \( x \) is small, we see that the linearization to (6) at \((0, 0)\) is
\[
\frac{dx}{dt} = v, \\
\frac{dv}{dt} = -(g/a)x.
\]
Solve this equation and find the approximate frequency of oscillation of the pendulum when the \( \theta \) is small.

Suppose now that we add a linear damping term to the pendulum equation; in other words, we assume that in addition to the gravitational force
\[ F_C = -mg \sin \theta, \]
the bob is subjected to a linear damping force
\[ F_{\text{damping}} = -ca \frac{d\theta}{dt}. \]
According to Newton’s law of motion,
\[ ma \frac{d^2 \theta}{dt^2} = \text{total force} = -ca \frac{d\theta}{dt} - mg \sin \theta, \]
and setting \( \theta = x \), we obtain the pendulum equation with linear damping
\[ \frac{d^2 x}{dt^2} + \frac{c}{m} \frac{dx}{dt} + \frac{g}{a} \sin x = 0, \tag{8} \]
where \( c, m, g \) and \( a \) are positive constants.

By reduction of order, we obtain the equivalent first order system:
\[
\frac{dx}{dt} = v, \\
\frac{dv}{dt} = -\left(\frac{g}{a}\right) \sin x - \left(\frac{c}{m}\right) v. \tag{9}
\]
The constant solutions are the simultaneous solutions to the system
\[
\begin{align*}
v & = 0, \\
-\left(\frac{g}{a}\right) \sin x - \left(\frac{c}{m}\right) v & = 0, \tag{10}
\end{align*}
\]
and just as in the case of the pendulum without damping, these constant solutions are
\[ (x, y) = (k\pi, 0), \quad \text{where } k \text{ is any integer.} \]

11. a. What is the linearization of (10)?

b. What are the eigenvalues of the linearization?

c. How does the linear damping term affect the stability of the constant solution \((0, 0)\)?
Comment: A somewhat more involved application of stability: Lagrange points

You are encouraged to look up the topic “Lagrange point” on Wikipedia, or do a google search. You will find that in gravitational systems, such as the earth-moon system, there are certain equilibrium positions in space at which gravitational forces are in stable equilibrium; this happens if the larger body has a mass which is more than 25 times as large as the mass of the smaller body. In the case of the earth-moon system, these two equilibria form equilateral triangles with the earth and the moon. In the case of the sun and Jupiter, Trojan asteroids orbit the Lagrange equilibria.

Stability of the equilibria involves a much deeper analysis than we have done in our examples, since one must use rotating coordinates. Nevertheless, at the end of the day, stability comes down to determine the eigenvalues of a certain matrix, just as in the examples we have been studying.