Our goal is to develop methods for solving linear systems of differential equations of the following type:

\[
\begin{align*}
\frac{dx_1}{dt} &= a_{11}x_1 + \ldots + a_{1n}x_n + f_1(t), \\
\vdots & \quad \vdots \\
\frac{dx_n}{dt} &= a_{n1}x_1 + \ldots + a_{nn}x_n + f_n(t).
\end{align*}
\] (1)

Here the \(a_{ij}\)'s are given constants and the \(f_i(t)\)'s are certain given functions. Systems of this type arise, for example, via Kirchhoff’s laws, in the theory of electrical circuits with linear circuit elements, as well as in many other applications (after possibly reducing order to eliminate any second-order derivatives which may occur). The first step is to find the general solution in the homogeneous case,

\[
\begin{align*}
\frac{dx_1}{dt} &= a_{11}x_1 + \ldots + a_{1n}x_n, \\
\vdots & \quad \vdots \\
\frac{dx_n}{dt} &= a_{n1}x_1 + \ldots + a_{nn}x_n.
\end{align*}
\] (2)

It turns out that once we have such a solution, we can find the general solution to the original system (1) by variation of parameters.

It simplifies matters considerably to rewrite the homogeneous system (2) in matrix form. If we let

\[
\begin{align*}
x &= \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \\
A &= \begin{pmatrix} a_{11} & \cdots & a_{1n} \\
& \ddots & \vdots \\
& \vdots & \ddots & \ddots \\
a_{n1} & \cdots & a_{nn} \end{pmatrix},
\end{align*}
\]

then the system (2) can be written quite economically as

\[
\frac{dx}{dt} = Ax.
\] (3)
A. Eigenvalues and eigenvectors

Solving systems like (3) efficiently requires ideas from linear algebra, including the important notions of eigenvalues and eigenvectors.

**Key Definitions.** An *eigenvalue* for the square matrix $A$ is a number $\lambda$ (possibly complex) such that

$$\det(A - \lambda I) = 0.$$ 

The *eigenspace* $V_\lambda$ for $A$ corresponding to the eigenvalue $\lambda$ is the set of solutions $b$ to the homogeneous linear system

$$(A - \lambda I)b = 0.$$ 

In symbols,

$$V_\lambda = \{b \in \mathbb{R}^n : (A - \lambda I)b = 0\}.$$ 

It is important to note that an eigenspace is a linear subspace of $\mathbb{R}^n$, as discussed in Math 3C. Finally, an *eigenvector* for $A$ corresponding to the eigenvalue $\lambda$ is a nonzero element $b \in \mathbb{R}^n$ such that $Ab = \lambda b$.

1. Suppose that $b$ is an eigenvector for the square matrix $A$ corresponding to the eigenvalue $\lambda$. Show that $x(t) = be^{\lambda t}$ is a solution to the homogeneous linear system (3).

2. a. Consider the homogeneous linear system

$$\begin{align*}
\frac{dx_1}{dt} &= 3x_1 + x_2 - 2x_3, \\
\frac{dx_2}{dt} &= 2x_2, \\
\frac{dx_3}{dt} &= 4x_1 + x_2 - 3x_3,
\end{align*}$$

which can be rewritten in matrix form as

$$\frac{dx}{dt} = Ax,$$

where

$$A = \begin{pmatrix} 3 & 1 & -2 \\ 0 & 2 & 0 \\ 4 & 1 & -3 \end{pmatrix}.$$ 

Find the eigenvalues $\lambda_1$, $\lambda_2$ and $\lambda_3$ of $A$.

b. Find eigenvectors $b_1$, $b_2$, and $b_3$ corresponding to the eigenvalues. Note that you have considerable choice in selecting eigenvectors.

c. Find the general solution to the homogeneous linear system (4).

Suppose that $A$ is an $(n \times n)$ square matrix. Then the equation

$$\det(A - \lambda I) = 0$$
is called the characteristic equation for \( A \). The characteristic equation for an \((n \times n)\)-matrix \( A \) is a polynomial equation of degree \( n \), a polynomial equation of the form

\[
a_n \lambda^n + \ldots + a_2 \lambda^2 + a_1 \lambda + a_0 = 0,
\]

For \( 2 \times 2 \) matrices, we can find the roots by using the quadratic formula:

\[
\lambda = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0a_2}}{2a_2}.
\]

If \( \lambda_1 \) and \( \lambda_2 \) are the roots to this equation, then

\[
a_2 \lambda^2 + a_1 \lambda + a_0 = a_2 (\lambda - \lambda_1)(\lambda - \lambda_2).
\]

For larger matrices, one may be able to find explicit solutions for the eigenvalues, but it often requires some luck. It is often necessary to use complex numbers to find the eigenvalues. But it is a remarkable and deep fact that complex numbers are sufficient for factoring polynomials of arbitrary degree:

**The Fundamental Theorem of Algebra.** Any polynomial of the form

\[
a_n \lambda^n + \ldots + a_2 \lambda^2 + a_1 \lambda + a_0,
\]

where the \( a_i \)’s are complex numbers can be factored into linear factors

\[
a_n \lambda^n + \ldots + a_2 \lambda^2 + a_1 \lambda + a_0 = a_n(\lambda - \lambda_1) \cdots (\lambda - \lambda_n),
\]

where \( \lambda_1, \ldots, \lambda_n \) are complex numbers.

This theorem is proven in more advanced courses on algebra. Although the Fundamental Theorem of Algebra guarantees that any polynomial can be factored into linear factors, it is often quite difficult to determine the factorization when the degree of the polynomial is large. In the sixteenth century, the Italian mathematicians Ferro, Tartaglia, and Cardan discovered a general method for finding roots to polynomials of the third degree, and shortly thereafter a method was discovered for treating polynomials of degree four.\(^1\) Later, the French mathematician Galois discovered that there are no general methods for finding roots of polynomials of degree five or greater.

Thus for linear systems of differential equations in more than three variables, one must often resort to numerical methods. The home page for the course includes a tutorial for using Mathematica which illustrates how one would calculate numerical solutions to homogeneous linear systems, such as (2).

**B. Matrices made up of eigenvectors**

For each of the eigenvalues \( \lambda_i \) for \( A \) the matrix \( A - \lambda_i I \) will be singular, and therefore by the theory of homogeneous linear systems of equations, there must exist a nonzero solution \( b_i \) to the matrix equation

\[
(A - \lambda_i I)b_i = 0.
\]

---

In other words, there is always a (nonzero) eigenvector \( \mathbf{b}_i \) corresponding to each eigenvalue \( \lambda_i \). One can show that if \( \lambda_1, \ldots, \lambda_n \) are distinct, then the corresponding eigenvectors \( (\mathbf{b}_1, \ldots, \mathbf{b}_n) \) are linearly independent. Thus the matrix

\[
B = (\mathbf{b}_1 \cdots \mathbf{b}_n)
\]

is invertible.

3. a. Suppose that we want to find a matrix \( B \) such that \( B^{-1}AB \) is diagonal, where

\[
A = \begin{pmatrix} 3 & 5 \\ 5 & 3 \end{pmatrix}.
\]

Carry out the first step by finding the eigenvalues \( \lambda_1 \) and \( \lambda_2 \) of \( A \).

b. Find the eigenspaces corresponding to each of these eigenvalues.

c. Find the corresponding eigenvectors \( \mathbf{b}_1 \) and \( \mathbf{b}_2 \).

d. Finally, put the eigenvectors \( \mathbf{b}_1 \) and \( \mathbf{b}_2 \) together as columns of a matrix \( B \) and calculate \( B^{-1} \).

e. Carry out the product \( B^{-1}AB \). How is it related to the eigenvalues?

**Theorem.** Suppose that the \( n \times n \)-matrix \( A \) has \( n \) distinct eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_n \), with corresponding eigenvectors \( \mathbf{b}_1, \ldots, \mathbf{b}_n \). Then the \( n \times n \)-matrix \( B \) whose columns are the eigenvectors \( \mathbf{b}_1, \ldots, \mathbf{b}_n \) satisfies the equation

\[
B^{-1}AB = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}.
\]

Although we are not concerned with giving formal proofs in this course, it is relatively easy to see why this theorem is true. By reducing to “column reduced echelon form,” we can show that linear independence of the eigenvectors \( \mathbf{b}_1, \ldots, \mathbf{b}_n \) ensures that the matrix \( B \) will be invertible. Let

\[
\mathbf{e}_i = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix},
\]

the 1 appearing in the \( i \)-th slot. Then multiplication by \( B \) takes the vector \( \mathbf{e}_i \) to \( \mathbf{b}_i \):

\[
\mathbf{e}_i \mapsto B\mathbf{e}_i = \mathbf{b}_i.
\]

On the other hand, the eigenvalue equation

\[
A\mathbf{b}_i = \lambda_i \mathbf{b}_i
\]
says that multiplication by $A$ takes $b_i$ to $\lambda_i b_i$:

$$b_i \mapsto A b_i = \lambda_i b_i.$$ 

Finally, $B^{-1}$ takes $b_i$ to $e_i$, and hence takes $\lambda_i b_i$ to $\lambda_i e_i$:

$$\lambda_i b_i \mapsto B^{-1} \lambda_i b_i = \lambda_i e_i.$$ 

Thus putting all of the multiplications together, we see that $B^{-1}A$ takes $e_i$ to $\lambda_i e_i$. But there is only one matrix with this property, namely,

$$B^{-1}A = \begin{pmatrix} \lambda_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \lambda_n \end{pmatrix}.$$ 

That is exactly what we wanted to prove.

4. a. Find the eigenvalues of

$$A = \begin{pmatrix} 1 & 4 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 5 & 2 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$ 

b. Find the eigenspaces corresponding to each of these eigenvalues.

c. Find a basis for $\mathbb{R}^4$ consisting of eigenvectors for $A$.

d. Find a $4 \times 4$-matrix $B$ such that $B^{-1}A$ is diagonal.

C. How matrices simplify solving systems of differential equations

We will now return to linear systems of differential equations of the following type:

$$\frac{dx_1}{dt} = a_{11} x_1 + \ldots + a_{1n} x_n + f_1(t),$$

$$\ldots$$

$$\frac{dx_n}{dt} = a_{n1} x_1 + \ldots + a_{nn} x_n + f_n(t).$$  \hspace{1cm} (6)

It will be quite convenient to rewrite this system by letting

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix}, \quad f(t) = \begin{pmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{pmatrix}.$$ 

Then (6) becomes

$$\frac{dx}{dt} = Ax + f(t).$$ \hspace{1cm} (7)
Our method for solving (7) is conceptually very simple. We introduce new variables \((y_1, \ldots, y_n)\), related to the old ones by a linear transformation

\[
\begin{align*}
x_1 &= b_{11}y_1 + \ldots + b_{1n}y_n, \\
x_n &= b_{n1}y_1 + \ldots + b_{nn}y_n,
\end{align*}
\]

where the \(b_{ij}\)'s are constants. We can write the equation of transformation in matrix form as

\[
x = By,
\]

where \(B = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\
\vdots & \ddots & \vdots \\
b_{n1} & \cdots & b_{nn} \end{pmatrix}\), and \(y = \begin{pmatrix} y_1 \\
y_n \end{pmatrix}\).

We will assume that \(B\) is invertible, so that

\[
y = B^{-1}x.
\]

Then equation (7) yields

\[
B \frac{dy}{dt} = \frac{dx}{dt} = Ax + f(t) = ABy + f(t).
\]

If we multiply through by \(B^{-1}\) this becomes

\[
\frac{dy}{dt} = B^{-1}ABy + B^{-1}f(t).
\]

Thus the system (7) can be put in the form

\[
\frac{dy}{dt} = (B^{-1}AB)y + g(t), \quad \text{where} \quad g(t) = B^{-1}f(t).
\]

The objective is to choose \(B\) so that \(B^{-1}AB\) is as simple as possible. In the previous section, we saw that if \(A\) has \(n\) distinct eigenvalues, it is possible to choose \(B\) so that \(B^{-1}AB\) is diagonal. In this case, the corresponding linear system will “decouple” into \(n\) equations which do not interact, a linear system of the form

\[
\begin{align*}
dy_1/dt &= \lambda_1y_1 + g_1(t), \\
\vdots \\
dy_n/dt &= \lambda_ny_n + g_n(t).
\end{align*}
\]

We can then solve each of these linear equations individually.

5. a. Consider the homogeneous linear system

\[
\begin{align*}
dx_1/dt &= 3x_1 + 5x_2, \\
dx_2/dt &= 5x_1 + 3x_2,
\end{align*}
\]

which can be written in the form

\[
A = \begin{pmatrix} 3 & 5 \\
5 & 3 \end{pmatrix}.
\]
Note that $A$ is the same matrix that you considered in problem 3. Using the eigenvectors you found when solving that problem, construct a matrix $B$ such that the change of variables

$$x_1 = b_{11}y_1 + b_{12}y_2,$$
$$x_2 = b_{21}y_1 + b_{22}y_2,$$

puts the system in diagonal form

$$\frac{dy_1}{dt} = \lambda_1 y_1,$$
$$\frac{dy_2}{dt} = \lambda_2 y_2,$$

and solve this system.

b. What is the general solution to the original system (9)?

**Note the format of the following worked out example:** Consider the homogeneous linear system

$$\frac{dx_1}{dt} = 3x_1 - 2x_2,$$
$$\frac{dx_2}{dt} = 4x_1 - 3x_2,$$  \hfill (10)

which can be rewritten in matrix form as

$$\frac{dx}{dt} = Ax, \quad \text{where} \quad A = \begin{pmatrix} 3 & -2 \\ 4 & -3 \end{pmatrix}.$$

The first step in solving this equation consists of finding the eigenvalues of the matrix $A$ by solving the characteristic equation

$$0 = \det(A - \lambda I) = \begin{vmatrix} 3 - \lambda & -2 \\ 4 & -3 - \lambda \end{vmatrix} = (3 - \lambda)(-3 - \lambda) + 8 = \ldots = \lambda^2 - 1.$$

We see that the eigenvalues are 1 and $-1$.

Next we need to find an eigenvector $b_1 = \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix}$ corresponding to the eigenvalue $\lambda_1 = 1$. We do this by finding a nonzero solution to the linear system

$$\begin{pmatrix} 3 - 1 & -2 \\ 4 & -3 - 1 \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix} = 0, \quad \text{or} \quad 2b_{11} - 2b_{21} = 0, \quad 4b_{11} - 4b_{21} = 0.$$

We can set $b_{11} = b_{21} = 1$, so that

$$b_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \text{and} \quad V_1 = \text{span} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Similarly, we need to find an eigenvector $b_2 = \begin{pmatrix} b_{12} \\ b_{22} \end{pmatrix}$.
corresponding to the eigenvalue \( \lambda_2 = -1 \). Its components must satisfy the linear system
\[
\begin{pmatrix}
3 + 1 & -2 \\
4 & -3 + 1
\end{pmatrix}
\begin{pmatrix}
b_{12} \\
b_{22}
\end{pmatrix} = 0, \quad \text{or} \quad 4b_{12} - 2b_{22} = 0, \quad 4b_{12} - 2b_{22} = 0.
\]
In this case, we can take \( b_{12} = 1 \) and \( b_{22} = 2 \), so that
\[
b_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \text{and} \quad V_2 = \text{span} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.
\]
We then set
\[
x = By, \quad \text{where} \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix},
\]
and the theorem of the preceding section implies that
\[
B^{-1}AB = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]
Thus in terms of the new variable
\[
y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}
\]
the system (10) becomes
\[
\frac{dy}{dt} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} y, \quad \text{or} \quad \frac{dy_1}{dt} = y_1, \quad \frac{dy_2}{dt} = -y_2.
\]
The general solution to this new system is
\[
y_1 = c_1e^t, \quad y_2 = c_2e^{-t}.
\]
Using the transformation (11), we see that the solution to the original system (10) is
\[
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} c_1e^t \\ c_2e^{-t} \end{pmatrix},
\]
or equivalently
\[
x_1 = c_1e^t + c_2e^{-t},
\]
\[
x_2 = c_1e^t + 2c_2e^{-t}.
\]
6. Use the above procedure to find the general solution to the linear system:
\[
\begin{aligned}
\frac{dx_1}{dt} &= 3x_1 - 2x_2 + e^{2t}, \\
\frac{dx_2}{dt} &= 4x_1 - 3x_2,
\end{aligned}
\]
which can be written in matrix form as
\[
\frac{dx}{dt} = Ax + f(t) \quad \text{where} \quad A = \begin{pmatrix} 3 & -2 \\ 4 & -3 \end{pmatrix} \quad \text{and} \quad f(t) = \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix}.
\]
Homework 5. Due Friday, February 5, 2010.

H.5.1.a. Find the matrix $A$ and vector-valued function $f(t)$ such that the linear system

$$\begin{align*}
\frac{dx_1}{dt} &= 4x_1 - 2x_2 + e^t, \\
\frac{dx_2}{dt} &= x_1 + x_2.
\end{align*}$$

assumes the matrix form

$$\frac{dx}{dt} = Ax + f(t).$$

b. Find the eigenvalues and eigenvectors for this matrix $A$.

c. Find a matrix $B$ so that $B^{-1}AB$ is diagonal.

d. Write the system of differential equations in the form

$$\frac{dy}{dt} = (B^{-1}AB)y + g(t),$$

and solve the resulting system.

e. Find the general solution to the original system of differential equations.