A. Review: bases for eigenspaces

We begin this project by reviewing some concepts you should have seen in Math 3C or Math 3CI.

**Abstract Definition.** A linear subspace of \( \mathbb{R}^n \) is a nonempty set \( V \) of vectors in \( \mathbb{R}^n \) such that whenever \( x \) and \( y \) are elements of \( V \), then the vectors \( x + y \) and \( cx \) are also members of \( V \), for any choice of real number \( c \).

Less formally, a linear subspace of \( \mathbb{R}^n \) is a plane of some dimension which passes through the origin. A key example is this: Suppose that \( A \) is an \( n \times n \) matrix with real entries and \( \lambda \) is a real number. Then

\[
V_\lambda = \{ b \in \mathbb{R}^n : Ab = \lambda b \}
\]

is a linear subspace of \( \mathbb{R}^n \), called the eigenspace for \( A \) for eigenvalue \( \lambda \). If \( V_\lambda \) is nonzero, then \( \lambda \) is called an eigenvalue for \( A \) and any nonzero element of \( V_\lambda \) is called an eigenvector for eigenvalue \( \lambda \).

**Definition.** Suppose that \( (b_1, \ldots, b_m) \) are elements of a linear subspace \( V \subseteq \mathbb{R}^n \). We say that \( (b_1, \ldots, b_n) \) span \( V \) and write

\[
V = \text{Span}(b_1, \ldots, b_m)
\]

if whenever \( x \in V \), then

\[
x = c_1 b_1 + \cdots + c_m b_m,
\]

for some choice of real numbers \( c_1, \ldots, c_m \).

1. If \( b_1 = (1, 0, 3) \) and \( b_2 = (0, 1, 1) \), then \( \text{Span}(b_1, b_2) \) is a plane in \( \mathbb{R}^3 \) which passes through the origin. Find an equation for that plane.
**Definition.** Suppose that \((b_1, \ldots, b_n)\) are elements of \(\mathbb{R}^n\). We say that these vectors are *linearly dependent* over \(\mathbb{R}\) if there exist real numbers \(c_1, c_2, \ldots, c_n\), not all zero, such that
\[c_1 b_1 + \cdots + c_m b_m = 0.\]

We say that a collection \((b_1, \ldots, b_n)\) of vectors are *linearly independent* over \(\mathbb{R}\) if they are not linearly dependent.

2. For which values of the real variable \(t\) are the vectors \(b_1 = (1, 0, 3, 2)\) and \(b_2 = (2, 0, 6, t)\) linearly dependent?

**Definition.** Suppose that \((b_1, \ldots, b_m)\) are elements of a linear subspace \(V \subseteq \mathbb{R}^n\). We say that \((b_1, \ldots, b_m)\) is a *basis* for \(V\) if
\[V = \text{Span}(b_1, \ldots, b_m)\] and \((b_1, \ldots, b_n)\) are linearly independent.

**Theorem.** Suppose that \(V \subseteq \mathbb{R}^n\) is the linear subspace of \(\mathbb{R}^n\) consisting of solutions to the homogeneous linear system
\[
\begin{align*}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0, \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0, \\
&\cdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0.
\end{align*}
\]

Then a *basis* for \(V\) is an ordered collection \((b_1, \ldots, b_m)\) of vectors in \(\mathbb{R}^n\) such that every solution \(x \in V\) can be written as
\[x = c_1 b_1 + \cdots + c_m b_m, \quad (1)\]
for a unique choice of real numbers \(c_1, \ldots, c_m\).

to find a basis for such a linear subspace \(V\) we use elementary row operations to put the coefficient matrix
\[
A = \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
& \cdots & \cdots & \cdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\]
into row reduced echelon form and then solve for the initial ones leading the rows to obtain a representation (1) for the space \(V\) of solutions. The properties of the row-reduced echelon matrix imply that this representation yields a basis.

For example, suppose that the row-reduced echelon matrix one obtains is
\[
\begin{pmatrix}
1 & -3 & 0 & -5 & 0 \\
0 & 0 & 1 & -7 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]
Then the corresponding system of equations is

\[
\begin{align*}
    x_1 - 3x_2 - 5x_4 &= 0, \\
    x_3 - 7x_4 &= 0, \\
    x_5 &= 0,
\end{align*}
\]

and the general solution to the system is

\[
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{pmatrix} = x_2 \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 5 \\ 0 \\ 7 \\ 1 \\ 0 \end{pmatrix},
\]

so a basis for \( V \) is

\[
\begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 5 \\ 0 \\ 7 \\ 1 \\ 0 \end{pmatrix}.
\]

We have seen that if \( A \) is an \( n \times n \) matrix with real entries, a useful strategy for solving the vector differential equation

\[
\frac{dx}{dt} = Ax
\]

involves finding the eigenvalues for \( A \) and then constructing a basis for each eigenspace.

3. Suppose that

\[
A = \begin{pmatrix}
-2 & 1 & 0 & 1 \\
1 & -2 & 1 & 0 \\
0 & 1 & -2 & 1 \\
1 & 0 & 1 & -2
\end{pmatrix}.
\]

This is the special case \( n = 4 \) of the matrix we considered in part B of Project 7 and its eigenvalues are 0, -2 and -4. Calculate a basis for each eigenspace directly, using the elementary row operations.

4. a. Suppose that

\[
A = \begin{pmatrix}
2 & 3 & 0 & 7 \\
0 & 2 & 0 & 5 \\
0 & 0 & 2 & 4 \\
0 & 0 & 0 & 2
\end{pmatrix}.
\]

The only eigenvalue for this matrix is 2. Find a basis for the eigenspace for eigenvalue 2.

b. Can you find a basis for \( \mathbb{R}^4 \) consisting of eigenvectors for \( A \)?
B. Repeated roots

It is an unfortunate fact of life that given a matrix $A$, it is not always possible to find an invertible matrix $B$ such that $B^{-1}AB$ is diagonal. The difficulty occurs in the “repeated root” case, the case where the characteristic equation

$$\det(A - \lambda I) = 0$$

has fewer than $n$ distinct solutions, when it may not be possible to find $n$ distinct eigenvectors to serve as columns of $B$. In linear algebra books, one can find a discussion of the best possible form for $B^{-1}AB$ under the heading, “Jordan canonical form.”

We content ourselves here with describing how to solve the linear system

$$\frac{dx}{dt} = Ax$$

in the repeated root case, leaving a discussion of the Jordan canonical form to more advanced courses. The idea behind the method is to find a basis for $\mathbb{R}^n$ consisting of generalized eigenvectors. By definition, a generalized eigenvector corresponding to the eigenvalue $\lambda$ is a solution $b$ to the homogeneous linear system

$$(A - \lambda I)^k b = 0.$$  

5. If $\lambda \in \mathbb{C}$, show that

$$\tilde{V}_\lambda = \{b \in \mathbb{C}^n : (A - \lambda I)^k b = 0\}$$

is a linear subspace of $\mathbb{C}^n$.

If it is nonzero, we call $\tilde{V}_\lambda$ a generalized eigenspace. Just like for ordinary eigenspaces, you can use the elementary row operations to construct a basis for each generalized eigenspace $\tilde{V}_\lambda$. It is proven in more advanced courses that an $n \times n$ matrix always possesses $n$ linearly independent generalized eigenvectors. If $B$ is an invertible matrix whose columns are generalized eigenvectors, it is not necessarily true that $B^{-1}AB$ is diagonal, but it is still possible to construct a general solution to the linear system (2) using generalized eigenvectors.

The idea is to make use of the matrix exponential. If $A$ is a square matrix, the matrix exponential $e^A$ is defined by substituting $x = A$ into the infinite series

$$e^x = 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4 + \ldots.$$  

The result is

$$e^A = 1 + A + \frac{1}{2!} A^2 + \frac{1}{3!} A^3 + \frac{1}{4!} A^4 + \ldots.$$  

For example, if

$$A = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix}$$
then one easily verifies that \( A^2 = 0 \), and hence
\[
e^A = I + A + 0 + \ldots = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 8 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 8 & 0 \end{pmatrix} + \ldots = \begin{pmatrix} 5 & -2 \\ 8 & -3 \end{pmatrix}.
\]
In this case, the matrix \( A \) is nilpotent, in accordance to the following definition.

**Definition.** An \( n \times n \) matrix \( A \) is said to be *nilpotent* if \( A^k = 0 \) for some integer \( k \).

6. Show that the matrix
\[
A = \begin{pmatrix} 0 & 2 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}
\]
is nilpotent, and calculate \( e^A \).

7. Show that the matrix
\[
A = \begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix},
\]
is not nilpotent. Nevertheless, one can calculate \( e^A \) in this case. What is the result? (You may need to remember the power series expressions for \( \sin t \) and \( \cos t \).)

The important fact about the matrix exponential is that
\[
x(t) = e^{tA}b
\]
is the solution to the initial value problem
\[
\frac{dx}{dt} = Ax, \quad x(0) = b. \tag{3}
\]
You can verify this by differentiating
\[
e^{tA} = 1 + tA + \frac{1}{2!} t^2 A^2 + \frac{1}{3!} t^3 A^3 + \frac{1}{4!} t^4 A^4 \ldots
\]
with respect to \( t \), thereby obtaining
\[
\frac{d}{dt} e^{tA} = A + \frac{2}{2!} t A^2 + \frac{3}{3!} t^2 A^3 + \frac{4}{4!} t^3 A^4 \ldots
\]
\[
= A + t A^2 + \frac{1}{2!} t^2 A^3 + \frac{1}{3!} t^3 A^4 + \ldots = Ae^{tA}.
\]
Thus
\[
\frac{d}{dt}(e^{tA}b) = e^{tA}b,
\]
and since \( e^{0A} = I \),
\[
x(t) = e^{tA}b
\]
is a solution to (3). We can summarize as follows:

**Theorem.** Suppose that $A$ is an $n \times n$ matrix with real entries. Then the solution to the initial value problem (3) is

$$x(t) = e^{tA}b,$$

where $e^{tA}$ is the matrix exponential.

This gives us another procedure (besides the eigenvalue-eigenvector method) for solving homogeneous linear systems of differential equations. Unfortunately, the process is complicated for general choice of $A$ and the power series solutions that one obtains might be hard to recognize. The advantage, however, is that it can be used to generate an approach for solving homogeneous linear systems in the case of repeated roots.

8. Use the Theorem and the result of problem 7 to find the solution to the initial-value problem

$$\frac{dx}{dt} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x, \quad x(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

9. Now let’s try a repeated root problem. Find the solution to the initial-value problem

$$\frac{dx}{dt} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x, \quad x(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

(Note that the $2 \times 2$ matrix in this case is nilpotent, so it is easy to calculate the matrix exponential.

10. a. Suppose that

$$\frac{dx}{dt} = Ax, \quad \text{where} \quad A = \frac{dx}{dt} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} x.$$

Although $A$ is not nilpotent in this case, $A - 2I$ is indeed nilpotent, so it is easy to calculate its matrix exponential. Show that

$$x = e^{2t} e^{(A - 2I)t} b$$

is a solution to the initial value problem (3).

b. Use (4) to find the solution to the initial-value problem

$$\frac{dx}{dt} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} x, \quad x(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

We are now ready to describe the general method for solving (2) in the repeated root case:
1. First, we need to find a basis for $\mathbb{R}^n$ consisting of generalized eigenvectors. (As mentioned before, we can use the elementary row operations to find a basis for each generalized eigenspace.)

2. For each eigenvector $b_i$ corresponding to eigenvalue $\lambda_i$, the solution to the initial value problem (3) with $b = b_i$ is just

$$x = e^{\lambda_i t}b_i.$$ 

3. For each generalized eigenvector for $\lambda_i$ which is not a genuine eigenvector, the solution to the initial value problem (3) with $b = b_i$ can be written in the form

$$x = e^{\lambda_i t}e^{t(A-\lambda_i I)b_i}.$$ 

Note that for some integer $k$, terms beyond the $k$-th in the matrix exponentials calculated in step 3 vanish, because

$$(A - \lambda_i I)^k b = 0,$$

and hence the infinite series solution collapses to manageable form.

4. The general solution to (2) is an arbitrary superposition of the solutions found in steps 2 and 3.

Unfortunately, the method can be somewhat lengthy to work out!

**Example.** To illustrate the method, we consider the linear system

$$\begin{align*}
\frac{dx_1}{dt} &= -x_1 - 4x_2, \\
\frac{dx_2}{dt} &= x_1 - 5x_2,
\end{align*}$$

or equivalently,

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}, \quad \text{where} \quad A = \begin{pmatrix} -1 & -4 \\ 1 & -5 \end{pmatrix}.$$ 

The characteristic equation is

$$0 = \det(A - \lambda I) = \begin{vmatrix} -1 - \lambda & -4 \\ 1 & -5 - \lambda \end{vmatrix} = (1 + \lambda)(5 + \lambda) + 4 = \lambda^2 + 6\lambda + 9.$$ 

The characteristic equation has the root $\lambda_1 = -3$, with multiplicity two. The corresponding eigenspace is the solution set to the linear system

$$\begin{pmatrix} -1 - (-3) & -4 \\ 1 & -5 - (-3) \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix} = 0, \quad \text{or} \quad 2b_{11} - 4b_{21} = 0, \quad 1b_{11} - 2b_{21} = 0.$$ 

The solution set to this system is

$$\text{span} \begin{pmatrix} 2 \\ 1 \end{pmatrix},$$
and a solution to the linear system is

\[ x = e^{-3t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \]

Since the eigenspace is one-dimensional, it is impossible to find two linearly independent eigenvectors for \( A \). In general, we would have to find a basis for each generalized eigenspace. In this case, however, all vectors in \( \mathbb{R}^2 \) must be generalized eigenvectors for the eigenvalue \(-3\). Thus we can use any element of \( \mathbb{R}^2 \) which is linearly independent from

\[ \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \]

as the additional generalized eigenvector. The method now says that

\[ x = e^{tA} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = e^{-3t} e^{t(A+3I)} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

is a solution. Note that

\[ e^{t(A+3I)} = I + t(A + 3I) + \frac{t^2}{2!}(A + 3I)^2 + \ldots \]

\[ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} + 0 + 0 + \ldots, \]

and hence

\[ e^{t(A+3I)} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 + 2t & -4t \\ t & 1 - 2t \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -4t \\ 1 - 2t \end{pmatrix}, \]

and a second solution to the linear system is

\[ x = e^{-3t} \begin{pmatrix} -4t \\ 1 - 2t \end{pmatrix}. \]

By the superposition principle, the general solution to the linear system is

\[ x = c_1 e^{-3t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} -4t \\ 1 - 2t \end{pmatrix}. \]

**Homework 8. Due Monday, March 1, 2010.**

H.8.1.a. Find the matrix \( A \) such that the linear system

\[ \begin{align*}
\frac{dx_1}{dt} &= 3x_1 - 2x_2, \\
\frac{dx_2}{dt} &= 8x_1 - 5x_2.
\end{align*} \]

assumes the matrix form

\[ \frac{dx}{dt} = Ax. \]
b. Find the eigenvalue $\lambda_1$ for this matrix $A$ and a corresponding eigenvector.

c. Find a generalized eigenvector which together with the eigenvector found in part b gives a basis for $\mathbb{R}^2$.

d. Find the general solution to the system of differential equations.