A. Constrained maxima and minima

One of the most useful applications of differential calculus of several variables is to the problem of finding maxima and minima of functions of several variables. The method of Lagrange multipliers finds maxima and minima of functions of several variables which are subject to constraints.

To understand the method in the simplest case, let us suppose that $S$ is a surface in $\mathbb{R}^3$ defined by the equation $\phi(x, y, z) = 0$, where $\phi$ is a smooth real-valued function of three variables. Let us assume, for the time being, that $S$ has no boundary and that the gradient $\nabla \phi$ is not zero at any point of $S$, so that the tangent plane to $S$ at a given point $(x_0, y_0, z_0)$ is given by the equation

$$(\nabla \phi)(x_0, y_0, z_0) \cdot \begin{pmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix} = 0.$$

**Problem 5.1.** Find an equation for the plane tangent to the surface

$$x^2 + 4y^2 + 9z^2 = 14$$

at the point $(1, 1, 1)$.

**Problem 5.2.** a. Sketch the level sets of the function $f(x, y) = x^2 + y^2$.
b. Sketch the curve $4x^2 + y^2 = 4$.
c. At which points of the curve $4x^2 + y^2 = 4$ does the function $f$ assume its maximum and minimum values?

Let $S$ be a surface in $\mathbb{R}^3$ defined by the equation $\phi(x, y, z) = 0$ and let $f(x, y, z)$ is a smooth function of three variables. Suppose that we are interested in finding the maximum and minimum values of $f$ on the surface $S$. Thus we seek to maximize or minimize $f$ subject to the constraint $\phi(x, y, z) = 0$. 


The function \( f \) determines a family of level surfaces \( f(x, y, z) = c \). If \((x_0, y_0, z_0)\) is a point on \( S \) at which a maximum or minimum value is attained, then \( S \) will be tangent to the level set

\[
f(x, y, z) = c, \quad \text{where} \quad c = f(x_0, y_0, z_0).
\]

Thus \((\nabla f)(x_0, y_0, z_0)\) must be a multiple of \((\nabla \phi)(x_0, y_0, z_0)\), i.e. there must exist a constant \( \lambda_0 \) such that

\[
(\nabla f)(x_0, y_0, z_0) = \lambda_0(\nabla \phi)(x_0, y_0, z_0).
\]

In other words, \((x_0, y_0, z_0, \lambda_0)\) must be a solution to the system

\[
(\nabla f)(x, y, z) = \lambda(\nabla \phi)(x, y, z), \quad \phi(x, y, z) = 0,
\]

or equivalently,

\[
\frac{\partial f}{\partial x} = \lambda \frac{\partial \phi}{\partial x}, \quad \frac{\partial f}{\partial y} = \lambda \frac{\partial \phi}{\partial y}, \quad \frac{\partial f}{\partial z} = \lambda \frac{\partial \phi}{\partial z}, \quad \phi = 0.
\]

In order to remember these systems more easily, we make use of the function

\[
H(x, y, z, \lambda) = f(x, y, z) - \lambda \phi(x, y, z).
\]

The solutions to either (1) or (2) are just the critical points of \( H \).

If \((x_0, y_0, z_0, \lambda_0)\) is a critical point for \( H \), we say that \((x_0, y_0, z_0)\) is a critical point for \( f \) on the surface \( S \) defined by the equation \( \phi = 0 \). These critical points are the candidates for the points on \( S \) at which \( f \) assumes its maximum and minimum values.

This process for finding the candidates for maxima and minima is called the method of Lagrange multipliers.

A theorem from analysis says that a continuous function always assumes its maximum and minimum values on any closed bounded smooth surface in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \).

**Problem 5.3.** Find the maximum and minimum values of the function \( f(x, y, z) = 4x + 2y + 6z \) on the ellipsoid \( S \) defined by the equation \( x^2 + 2y^2 + 3z^2 - 1 = 0 \).

**Problem 5.4.** a. Suppose that we want to find the maximum and minimum values for the function \( f(x, y, z) = xyz \) on the sphere \( x^2 + y^2 + z^2 = 1 \). In this case, the constraint equation is

\[
\phi(x, y, z) = x^2 + y^2 + z^2 - 1 = 0,
\]

and the function \( H \) is given by the formula

\[
H(x, y, z, \lambda) = xyz - \lambda(x^2 + y^2 + z^2 - 1).
\]

Find the critical points of \( H \).

b. Use the critical points you found in part a to find the maximum and minimum values for \( f(x, y, z) = xyz \) on the sphere \( x^2 + y^2 + z^2 = 1 \).
**Problem 5.5.** Here is a more “practical” application of Lagrange multipliers. Suppose that we want to construct a box, open on the top, of given volume, say 32 cubic inches, utilizing a minimal amount of cardboard. Let

\[ x = \text{width of box}, \quad y = \text{length of box}, \quad z = \text{height of box}. \]

Then the area of the base is \( xy \), while the area of the four sides is \( 2xz + 2yz \), so the total area of cardboard used is

\[ f(x, y, z) = xy + 2xz + 2yz. \]

We need to minimize this function \( f \) subject to the constraint

\[ \text{volume of the box} = xyz = 32. \]

Our constraint function in this case can be taken to be

\[ \phi(x, y, z) = xyz - 32. \]

Find the dimensions of the box which uses the least amount of cardboard.

The method of Lagrange multipliers extends to functions of \( n \) variables in the obvious way. If we want to maximize the function \( f(x_1, \ldots, x_n) \) on the hypersurface \( S \) represented by the constraint equation \( \phi(x_1, \ldots, x_n) = 0 \), where \( \nabla \phi \neq 0 \) at every point of surface \( S \), we form the function

\[ H(x_1, \ldots, x_n, \lambda) = f(x_1, \ldots, x_n) - \lambda \phi(x_1, \ldots, x_n). \]

If \( c = (c_1, \ldots, c_n) \) is a point on \( S \) at which \( f \) assumes its maximum or minimum value, there is a real number \( \lambda_0 \) such that \( (c_1, \ldots, c_n, \lambda_0) \) is a critical point for the function \( H \).

**B. Parametrizations of curves**

Suppose that \( C \) is a curve in \( \mathbb{R}^n \). A **parametrization** of \( C \) is a vector-valued function \( x : \mathbb{R} \to \mathbb{R}^n \) or \( x : [a, b] \to \mathbb{R}^n \), where \( [a, b] = \{ t : a \leq t \leq b \} \), such that \( x(t) \) traces out \( C \) exactly once. (In some cases we will allow the endpoints of \( C \) to be covered twice.) We write

\[ x(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \]

and call the scalar-valued functions \( x_1(t), \ldots, x_n(t) \) the **components** of \( x(t) \). A **parametrized curve** is a curve together with a parametrization of that curve.

For example, suppose that \( C \) is the line in \( \mathbb{R}^n \) which passes through the point \( p = (p_1, \ldots, p_n) \) and is parallel to the vector \( v = (v_1, \ldots, v_n) \). A parametrization of \( C \) is

\[ x : \mathbb{R} \to \mathbb{R}^n, \quad \text{defined by} \quad x(t) = p + tv. \]
If \( C \) is the line segment in \( \mathbb{R}^n \) from the point \( p = (p_1, \ldots, p_n) \) to the point \( q = (q_1, \ldots, q_n) \), we let \( v = q - p \), and parametrize \( C \) by
\[
x : [0, 1] \to \mathbb{R}^n \quad \text{where} \quad x(t) = p + tv.
\]

**Problem 5.6.**

a. Find a parametrization for the straight line segment in \( \mathbb{R}^3 \) from the point \( (1, 0, 3) \) to the point \( (1, 1, 1) \).

b. Find a parametrized curve that might represent the trajectory of a particle in \( \mathbb{R}^3 \) which starts at the point \( (1, 0, 3) \) when \( t = 0 \) and has velocity \( v = (3, 6, 1) \). Suppose that we take \( C \) to be a circle of radius \( a \) centered at the origin and lying in a plane spanned by two unit-length vectors \( e_1 \) and \( e_2 \), which are perpendicular to each other. For an arbitrary choice of \( t \), the point
\[
x(t) = a(\cos t)e_1 + a(\sin t)e_2
\]
lies on \( C \). As \( t \) ranges over the reals, \( x(t) \) traces out the circle infinitely many times. If we want to cover the circle just once, we can restrict \( t \) to range over the interval \( [0, 2\pi] \). The circle of radius \( a \) centered at the point \( p \) and lying in the plane spanned by the two perpendicular unit-length vectors \( e_1 \) and \( e_2 \) is parametrized by
\[
x : [0, 2\pi) \to \mathbb{R}^n \quad \text{where} \quad x(t) = p + a \cos t e_1 + a \sin t e_2.
\]

**Problem 5.7.**

a. Find a parametrization for the circle of radius 3 in \( \mathbb{R}^2 \) with center at the point \( (1, 2) \).

b. Find a parametrization for the circle of radius 3 in \( \mathbb{R}^3 \) with center at the origin which lies in the plane \( x + y + z = 0 \). (Hint: Use the cross product to find two unit-length vectors in the plane which are perpendicular to each other.

**Problem 5.8.** Consider the ellipse \( x^2 + 2xy + 2y^2 = 1 \) in \( \mathbb{R}^2 \). To parametrize this curve, we first write it in the form \( u^2 + v^2 = 1 \), where
\[
\begin{align*}
u &= x + y, & u &= x - v, \\
v &= y, & y &= v.
\end{align*}
\]
Use this transformation to construct a parametrization of the ellipse.

Clearly a parametrization of a curve gives more information than just what the curve is. For example, one of Kepler’s laws asserts that the planets move around the sun in ellipses. But simply knowing that a planet is moving along a given ellipse does not give a complete description of its motion. We need to know not just the curve itself, but also how the curve is parametrized, in order to know where the planet is at each point in time.

**Homework 5.** Due Friday, April 30, 2010.

**H.5.1.** Suppose that \( S \) is the unit sphere defined by the equation
\[
\phi(x_1, \ldots, x_n) = x_1^2 + \ldots + x_n^2 - 1 = 0,
\]
and that

\[ f(x_1, \ldots, x_n) = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \cdots \\ x_n \end{pmatrix} = x^T A x, \]

where \( A \) is a symmetric \( n \times n \) matrix \((a_{ij} = a_{ji})\) and

\[ x = \begin{pmatrix} x_1 \\ \cdots \\ x_n \end{pmatrix}. \]

Let

\[ H(x, \lambda) = x^T A x - \lambda \phi(x). \]

Show that each critical point \((x, \lambda)\) of \( H \) consists of an eigenvector for \( A \) with eigenvalue \( \lambda \), the eigenvector having unit length.